Supplier-Manufacturer Coordination in Capacitated Two-Stage Supply Chains

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ABSTRACT

Manufacture-to-order is an increasingly popular strategy in commodity electronics and other similar markets where many different product configurations can be produced from common components. To succeed in this environment, manufacturers need to keep both cost and order fulfillment time low. In this article, we compare three different mechanisms that a manufacturer, whose revenues depend on order delays, may use to affect its component supplier’s inventory decisions. These mechanisms are specifying components inventory level, offering a share of the earned revenues to the supplier (called simple revenue sharing), and offering a two-part revenue-sharing scheme. We show that whereas the first two approaches do not lead to supply chain coordination, the two-part scheme does. We demonstrate with numerical experiments that up to a point, the component supplier benefits from having a high utilization of its production facility, whereas the manufacturer benefits from having excess production capacity.

Keywords: Game Theory, Inventory, Supply chain coordination, Incentives, Production economics.

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1 Introduction

The ability to quickly assemble and deliver custom products is a winning competitive strategy; customers get what they want and the manufacturer avoids the costs of shortages and overages (Serwer, 2002). Those who succeed in this business need to have short order-turnaround times and cost-efficient production methods. A common practice by manufacturers (also called buyers hereafter) is to require their suppliers to keep ample inventories of components; often as a condition for winning the supply contract. This practice helps reduce order fulfillment time and may lower cost to buyers since investment in each unit of supplier-held inventory is at the marginal cost of production, rather than being at the higher wholesale price charged to buyers. A well-known and successful model of this type of manufacturing operations is the Dell Computer Corporation. It outsources the manufacture of many components and relies on quick deliveries from suppliers to keep order turnaround times low; reported to be about 18 days (Anonymous, 2000).

Naturally, it is appropriate to ask whether the practice of mandating a certain level of components stock is optimal for the supply chain? How can a manufacturer leverage the supply contract with its key components supplier to achieve faster order fulfillment and cost-efficient production? Which types of contracts are Pareto and channel optimal? We attempt to answer such questions in this article.

A manufacturer’s demand rate and the sensitivity of its revenues to actual delivery times can vary significantly depending upon its choices regarding positioning, market characteristics, and competitors’ responses. Although our models focus on bilateral (supplier-manufacturer) interactions, the factors mentioned above make the manufacturer’s sales revenues negatively correlated with order fulfillment times. The exact nature of this dependence can vary from situation to situation. For example, the manufacturer may incur a tardiness penalty based on the actual delivery time, or else it may need to pay for expedited shipping when orders are delayed (see Section 3 for more details). It should not come as surprise that the simultaneous management of lead times and inventories (particularly component inventories) has attracted considerable attention from industries that are moving toward a manufacture- or configure-to-order strategy (for example, see Ervolina and Katircioglu, 2000; and Yao et al., 2000). Our focus is primarily on how supply contracts can be used by the manufacturer to support its strategic choices. That is, our models treat the choice of the size of the product portfolio and the promised speed of delivery as exogenous.

Specifically, in this paper we investigate the interaction between a manufacturer (M) and its component supplier (S) under two situations: central-planner and decentralized decision-making.
The central planner model acts primarily as a benchmark against which the more common decentralized supply chain design can be compared. Our approach is to first analyze the simpler case in which the manufacturer’s revenue depends on the actual delivery time for each item. Later, we expand our results in two directions by modeling other types of revenue functions, and by relaxing certain assumptions about the characteristics of the supply system.

As mentioned earlier, we have anecdotal evidence which suggests that in several industries it is common for M’s purchasing managers to require S to maintain a minimum level of inventory of ready-to-ship components, from which M can draw supplies as demand unfolds. We assume that the supplier attempts to keep a stable stock of components by performing an item-for-item replenishment. In the inventory literature, this type of inventory control policy is called the base-stock policy. Mandatory stock keeping increases S's costs, which in the long run are passed on to M in the form of a higher per-unit wholesale price. We show later in this article that when the wholesale price is based on cost plus a fixed markup, which is a common industry practice, mandatory stock keeping is suboptimal. In other words, this arrangement fails to deliver channel coordination. Throughout this article, we assume that complete information is available to both players about each other’s costs and about the characteristics of each other’s production systems.

We propose a benchmark revenue sharing scheme in which M first declares its intention to share a fraction $\alpha$ of its realized revenues with S. This creates an incentive for S to maintain some inventory of components since that reduces order fulfillment time, which in turn increases per-unit revenue. The supplier trades off the benefits of receiving a fraction of the higher per-unit revenue that results from larger components inventory against the increased holding costs, and responds by choosing the target inventory level of components that maximizes its profit. We show that for each offered revenue-fraction, there exists a unique profit-maximizing target stock level for the supplier. Each pair of revenue-fraction and the corresponding target stocking level is called a contract. Only those combinations that result in positive expected payoffs to both decision makers constitute the feasible set of individual strategies. Since M can also compute S’s response, it picks the individually-optimal shared revenue-fraction. We refer to this contract as the Stackelberg equilibrium contract and use it as a benchmark against which we compare other decentralized schemes (see the last paragraph of Section 4.2 for reasons why we choose this benchmark).

Next, we show that the benchmark equilibrium contract does not achieve coordination. Therefore, we also propose a continuum of two-part revenue sharing contracts each of which induces S to choose the overall optimal target inventory level, denoted by $b_0$. Here, M presents two choices to S. The first option promises S a revenue fraction $\frac{\alpha}{\alpha}$ and allows S to pick any target stock level.
The second option offers a greater revenue fraction $\overline{\alpha}$, but this option is available only if $S$ chooses a base-stock level $b \geq b_0$. Thus, $M$ specifies $\alpha$, $\overline{\alpha}$, and $b_0$. The parameter $\overline{\alpha}$ is not unique. Instead, its value must lie in a range to ensure participation by both players.

It can be shown that the contract described above is an instance of contracts involving a transfer payment from $M$ to $S$ with two parts: a linear positive part comprising of a fraction of realized revenue, and a nonlinear part comprising of either a penalty or a holding cost subsidy that depends on the average delay in sourcing components from the supplier. The delay-based penalty (holding cost subsidy) is positive only if average delay exceeds (is smaller than) a critical level and this amount is subtracted from (added to) the supplier’s share of realized revenue before the supplier is paid. Since in our models, the average delay in procuring a component is affected only by the supplier’s stock level, the critical level of delay can be set such that the penalty (subsidy) goes to zero if $S$ picks a stock level greater than (smaller than) $b_0$. Both players benefit from participating in this scheme in the sense that their payoffs are at least as much as they would receive under the equilibrium contract. For each contract of this kind, the total excess profit (defined as the difference between total supply chain expected profit under a central planner and the sum of $S$ and $M$ expected profits under the equilibrium contract) is fixed. However, it can be divided in different ways between the two players depending on the value of $\overline{\alpha}$, which may in turn depend on the relative bargaining power of the two players.

We present several numerical examples to study how the size of $S$’s and $M$’s production capacity, and unit production costs, affect their individual and joint welfare. Our analysis reveals that up to a point, $S$’s expected profit under the Stackelberg equilibrium solution may increase with increasing utilization of its resources, even though the overall supply chain profit declines. This means that a supplier whose order book is relatively full is more likely to extract greater rents from the manufacturer for quick delivery of components. Eventually, as $S$’s utilization approaches 1, its profits also begin to decline. In practice, we expect that this phenomenon will happen over a relatively small range of values of $S$’s utilization. If $S$’s production facility becomes too congested, it will no longer be competitive as compared to other potential suppliers. That is, the supplier should choose to operate with a relatively high utilization of its resources without losing competitiveness. In contrast, our examples show that $M$ should maintain ample excess capacity since its expected profits decline sharply when its capacity utilization is high.

This article is organized as follows. We provide a review of relevant literature in Section 2. Section 3 describes the model, notation, and basic performance measures. It also includes the benchmark central-planner model. The three models reported in Section 4 deal respectively with
the fixed-markup, simple revenue-sharing, and the two-part revenue-sharing contracts. Section 5 contains numerical examples and insights. Extensions of the base-case scenario can be found in Sections 6 and 7.

2 Literature Review

Supply contracts have been studied extensively in recent literature as possible mechanisms for achieving coordination. Coordination is realized when each decision maker in the supply chain, acting rationally, makes decisions that are optimal for the supply chain as a whole. Issues addressed by this literature range from the design of supply contracts (type of contracts, specifying parameters within a particular type of contract) to the impact of information asymmetry. A recent volume edited by Tayur, Ganeshan and Magazine (1999) contains four expository articles that provide a thorough review and synthesis of the emerging literature in this area (Anupindi and Bassok, 1999; Corbett and Tang, 1999; Lariviere, 1999; and Tsay et al., 1999). A more recent review can be found in Cachon (2004).

Most supply contracts contain some elements of risk- and reward-sharing between suppliers and retailers. This creates incentives for all decision makers to choose options that benefit the supply chain as a whole. For example, by offering a buy back clause when the retailer faces uncertain demand for fashion goods, the supplier shares overage risk with the retailer. This creates incentives for the retailer to stock more, thereby increasing expected sales. In absence of such risk sharing, it has been shown that a rational retailer will choose to stock less than the channel-optimal quantity. The literature demonstrates that many different contract parameters can be used to achieve coordination; some examples are quantity discounts, returns (buy backs), quantity flexibility, and the use of subsidies/penalties.

Nearly all of the earlier marketing and operations management studies take the view that the supplier (stage-1 decision maker) needs to create incentives for the retailer (stage-2 decision maker) to maintain the right amount of inventory in order to achieve the highest possible supply chain profits (see Cachon and Lariviere, 2001, for an exception in which the retailer acts as the leader). Important contributions to this literature can be found in Jeuland and Shugan (1983), Pasternack (1985), Moorthy (1987), Kandel (1996), Narayanan and Raman (1997), and Emmons and Gilbert (1998). In contrast, in our model, the manufacturer (equivalent to the retailer in other models) keeps no inventory and tries to create incentives for the supplier to keep the channel-optimal level of components inventory upstream in the supply chain. In this sense, our approach represents interactions between the decision makers of a pull system, whereas the earlier studies have focused
on push systems. Furthermore, situations in which revenues depend on lead times are not considered by others.

Articles that model supply operations fall into two broad categories. In the first category are newsvendor type models suitable for items with a short selling season. Lee, Padmanabhan, Taylor, and Whang (2000) study price protection in the personal computer business. Price protection is used as a form of subsidy offered for unsold inventory when the price drops during the life of a product. This paper shows that supply chain coordination can be achieved via the price protection mechanism in markets with a fast rate of product obsolescence. Donohue (2000) shows that a supply system can be coordinated when the retailer is allowed to place a second order at a later time after more accurate demand information has been acquired. The supplier charges a higher wholesale price for the second order which is supplied at shorter notice. The contract parameters are the two different wholesale prices (one for each order) and a returns price. Eppen and Iyer (1997) study supply agreements in the fashion industry by allowing a two-transactions sales mode similar to that used in Donohue (2000).

Lately, the problem of coordinating supply chains with perpetual demand have also attracted more attention. Lee and Whang (1999) study two-stage single agent models (similar to the Clark and Scarf, 1960 inventory model) in which only the upper stage incurs a holding cost while the lower stage incurs a backorder penalty. Lee and Whang propose a non-linear transfer payment contract to align the agents by utilizing an echelon inventory policy. Chen (1999) studies a game analogous to the popular Beer game (Sterman, 1989), except that demand is composed of independent random variables with a common distribution known to all players. Chen proposes a linear incentive alignment scheme based on accounting inventory level instead of actual inventory level. Accounting inventory consists of actual inventory level plus the inventory on order. Cachon and Zipkin (1999) propose a linear transfer payment contract in which the transfer payment parameters depend upon the system performance. The transfer payments ensure that each firm’s equilibrium decisions coincide with the overall supply chain optimal decisions.

Two studies require special attention on account of having a close relationship with our models. The first study is by Cachon and Lariviere (2000) in which the authors discuss the strengths and weaknesses of revenue sharing contracts. They show that revenue sharing contracts can coordinate supply systems in many commonly-studied scenarios, including the case when demand is price-dependent. Since we too focus on revenue sharing contracts, many of their observations also apply to our models. Note, however, that they are concerned with supplier-retailer relationships and on modeling items with a short selling season. Revenue sharing contracts continue to coordinate the
supply chain in all problem instances we analyze. This includes the case where the fraction of demand met depends on the level of inventory.

The second study is by Caldentey and Wein (2003) who use the linear transfer payment contract, similar to Cachon and Zipkin (1999) and further extend its use to align a two-stage supply chain. Both Caldentey and Wein (2003) and this article treat the first stage (supplier’s) operations as a $M/M/1$ queue. However, there are significant differences in the two approaches. In our model, some processing is required at the manufacturer’s facility before the product is ready for the customer. Caldentey and Wein’s second stage player is a retailer who stocks finished products that require no further processing. We also incorporate congestion effects at the manufacturer’s production facility by modeling it as a queue. Caldentey and Wein assume that the retailer earns a fixed revenue per unit sold and that backorder allocation fraction is exogenously determined. In contrast, we model the situation in which revenue per unit is lead-time sensitive and coordination is achieved by adjusting the share of the revenue that each player receives. Since longer lead times result in smaller per-unit revenue, in effect the two players share a lateness penalty. The key difference is that the manner in which the lateness penalty is shared by the two players is determined by the outcome of their game. It is not an exogenous parameter as assumed by Caldentey and Wein.

3 The Model

A schematic of the supply chain under investigation is shown in Figure 1. In this schematic, the flow of materials is indicated by solid arrows whereas the flow of information is shown by arrows with a dash-pattern. All customer orders experience a processing delay at the manufacturer’s production facility where products are customized to meet order specifications. As soon as an order is received, a raw-material kit is immediately released to the supplier’s production facility. If component kits are in stock, a kit is also immediately sent to the manufacturer where it joins the queue of orders waiting to be processed. Otherwise, the customer order is backlogged until a components’ kit becomes available. All orders are processed on a first-in-first-out basis at both production facilities. Thus, in addition to the processing delay at the manufacturer, an order may experience delays caused by the stockout of components and due to congestion at the manufacturer’s production facility.

There are several different ways in which the manufacturer’s revenue may depend on the delivery delay. For example, when M supplies repeat orders from a single customer, the per-unit revenue could be a function of the realized delay in fulfilling an order, or else it could depend on M’s performance in terms of the long-run average order fulfillment time. Such situations often arise as
a consequence of the terms of the supply contract between \( M \) and its customer. In the direct-to-market or retail environment, \( M \) is more likely to experience either loss of sale, or reduced revenue if the time to fill an order exceeds the promised (or market-norm) delivery time. The lower net revenues can result either from the additional cost of expedited shipping, or from a price discount that may be necessary to retain the customer. The value of the promised delivery time and the associated price-discount is determined by market characteristics and by \( M \)’s positioning relative to its competitors. They are not decision variables in this article. We present models for all of these different types of dependencies; starting first with a detailed analysis of the base case in which revenue is a linearly decreasing function of realized delay, and both \( S \) and \( M \) are risk-neutral rational decision makers. Notice that under these circumstances, expected profits of \( S \) and \( M \) are the same irrespective of whether per unit revenue depends on realized delay or on average delay.

In order to further formalize our models, we use the following notation and assumptions. Some of these assumptions, such as Poisson order arrivals and exponential processing times are made for mathematical tractability and to demonstrate a constructive approach to modeling replenishment operations. Many of these assumptions are relaxed in Section 7.

\[
\begin{align*}
\lambda &= \text{Demand arrival rate (demand is stationary and Poisson).} \\
\mu_i &= \text{Production capacity (rate) of player-}i \ (1 = S \text{ and } 2 = M), \text{ exponential processing times.} \\
\rho_i &= \text{Player-}i \text{ capacity utilization.} \\
b &= \text{Target level of components inventory.} \\
L_i(b) &= \text{Delivery delay due to player-}i. \ L = L_1 + L_2. \\
c_i &= \text{Unit production cost for player-}i. \\
\pi(b) &= \text{Average revenue from the sale of a unit of finished product.} \\
h &= \text{Inventory holding cost (dollars per unit per unit time).} \\
I(b) &= \text{Number of components in stock at an arbitrary observation epoch.}
\end{align*}
\]

Note that we are explicitly marking dependence of \( \pi, L_i \) and \( I \) on \( b \), which is the key decision variable in our models. In the sequel, averages are denoted with a bar notation.

We consider three possibilities in this article. In the first model, which we also call the base-case
model, the average per-unit revenue is assumed to be a linear function of realized (or average) lead
time. That is:

\[ \pi(b) = E[p_0 - \beta L(b)] = p_0 - \beta \bar{L}(b). \]  (1)

The terms \( p_0 > 0 \) and \( \beta \geq 0 \) are parameters of the model. A detailed analysis of the base-case
model can be found in Sections 4 and 5. Our second model is that of a direct-to-market firm that
offers a uniform delivery guarantee of \( \ell \) time periods to all its customers. When it cannot meet
this promise, we assume that it incurs a fixed expedited shipping charge. Specifically, the average
per-unit revenue is the following step-function:

\[ \pi(b) = r_1 \text{Prob}(L(b) \leq \ell) + r_2 \text{Prob}(L(b) > \ell), \]  (2)

where \( r_1 > r_2 \). Finally, the third type of revenue function, as explained below, models lost sales.

Suppose the customers are impatient and choose to buy from a competitor if the manufac-
turer cannot supply within a market-standard lead-time. Since each product is custom made,
the manufacturer must be able to predict production delays accurately in order to avoid having
a custom-built product on its hands that is no longer needed by the customer who ordered it.
Without this requirement, the manufacturer’s business model becomes questionable. We assume
that if the manufacturer finds the components in stock when it receives the order, it can supply the
product on time almost certainly. That is, its own assembly time is predictable and relatively short
as compared to customer expectation. On the other hand, when the component is not in stock, the
order delays are highly unpredictable. We therefore assume that in that case the customer does not
place the order. Through first-hand experience, we have found that some direct-to-market compa-
nies offer to substitute the customer’s desired configuration by an equivalent configuration for which
components are available, or quote longer than the standard delivery delays when components are
not in stock. Our model can be viewed as an abstraction of this practice where customers walk
if delivery cannot be guaranteed due to component shortage. The situation in which a fraction of
sales are lost due to tardiness and the remaining customers place orders anyway can be handled in
a similar way. The per-unit revenue is a function of the component inventory level and we use the
following functional form:

\[ \pi(b) = r \text{Prob}(I(b) > 0). \]  (3)

In equations (1), (2) and (3), the dependence of \( \pi \) on other parameters that determine \( L \) and \( I \)
(e.g., the \( \rho_i \)'s) is suppressed since those are not decision variables in our models.

Exact expressions for the performance measures of the two-stage supply chain shown in Figure
1 can be obtained only in certain special cases. These cases arise when either \( b = 0 \) or \( b = \infty \) (see
Lee and Zipkin, 1992 for details). The main difficulty is that the arrival of component kits to M’s production facility is not a renewal process. In fact, the inter-arrival times are correlated through their dependence on the number of orders that are yet to be processed by the supplier. Lee and Zipkin (1992) propose a heuristic procedure, which in our situation will amount to assuming that the delay in the second stage is exponentially distributed with parameter $\mu_2(1 - \rho_2)$. Note that the exact same distribution of $L_2$ is realized when stage-2 is assumed to behave like a $M/M/1$ queue.

Buzacott, Price and Shanthikumar (1992) also study the two-stage production system shown in Figure 1. They characterize the inter-arrival time distribution to stage-2, and show that its squared coefficient of variation is between 0.8 and 1. Both articles also provide numerical examples which confirm that approximating the distribution of $L_2$ by the distribution of delay in a $M/M/1$ queue is a reasonable approximation. Gupta and Selvaraju (2004) show that the $M/M/1$ approximation provides an upper bound on the true value of the mean delay experienced by a customer order. This implies that when the revenue function is linear in order delay, the true expected profit experienced by both parties will be somewhat better than what is predicted by our model. Therefore, contracts that are formed on the basis of this approximation will be always feasible. For these reasons, we adopt this approximation to obtain performance measures. Later, we show that as long as $\pi(b)$ is increasing concave in $b$, all our results continue to hold.

Using the approximation described above, the following expressions for the various measures of performance can be worked out (Proofs are omitted in the interest of brevity):

$$\bar{I}(b) = b - \frac{\rho_1[1 - \rho_1^b]}{1 - \rho_1}.$$ (4)

$$\bar{L}_1(b) = \frac{\rho_1^b}{\mu_1(1 - \rho_1)}.$$ (5)

$$\bar{L}_2 = \frac{1}{\mu_2(1 - \rho_2)}.$$ (6)

Clearly, $\bar{L}(b) = \frac{\rho_1^b}{\mu_1(1 - \rho_1)} + \frac{1}{\mu_2(1 - \rho_2)}$ is decreasing in $b$ and increasing in $\rho$. Similarly, we can show that $\bar{I}$ is increasing in $b$, and decreasing in $\rho_1$. The minimum and maximum values of average delivery delay can also be computed. These turn out to be: $\bar{L}_{\text{min}} = \bar{L}(\infty) = \frac{1}{\mu_2(1 - \rho_2)}$ and $\bar{L}_{\text{max}} = \bar{L}(0) = \frac{1}{\mu_1(1 - \rho_1)} + \frac{1}{\mu_2(1 - \rho_2)}$. Note that the calculation of $\bar{L}(b)$, the key performance measure that determines $\pi(b)$ in the base-case (see equation 1), does not require that the supplier’s workload come from a single source: the manufacturer. In fact, as long as we continue to use Lee and Zipkin’s approximate model of an exogenous supplier, the load on its production facilities can come from multiple sources of demand. In other words, we can use $\rho_1$ as a measure of the degree to which the supplier’s order book is full, where a portion of this load comes from the
manufacturer. Performance of different supply contracts when supplier faces different degrees of workload is presented in Section 5.

3.1 The Central-Planner Model

Let \( z_0 \) denote the supply chain expected profit function when all production and inventory decisions are made by a single decision maker. Then, it is easy to see that:

\[
z_0(b) = \lambda \pi(b) - h \bar{I}(b) - \lambda(c_1 + c_2). \tag{7}
\]

At first our focus is only on the per-unit revenue function of the type shown in equation (1). Other types of relationships are dealt with in Section 6. It is straightforward to verify from (4) – (6) that \( \bar{I}(b) \) and \( \bar{L}(b) \) are both convex in \( b \). Therefore \( \pi(b) = p_0 - \beta \bar{L}(b) \) is a concave function of \( b \). That immediately leads to the conclusion that \( z_0 \) is concave in \( b \); being the sum of concave functions.

In order to avoid trivial situations, we assume that revenue earned per unit sold exceeds the combined production costs of M and S, at least when the average lead time is at its smallest value, that is \( \pi(\infty) > c_1 + c_2 \). This inequality can be simplified to provide an upper bound on the maximum permissible value of \( \rho_2 \):

\[
\rho_2 < \frac{\lambda(p_0 - (c_1 + c_2))}{\beta + \lambda(p_0 - (c_1 + c_2))}. \tag{8}
\]

If \( z_0(b) \) is decreasing in \( b \) at \( b = 0 \), then it follows from the concavity of \( z_0(\cdot) \) that it must be decreasing in \( b \) for every \( b \in [0, \infty) \). Therefore, in such cases the optimal base-stock level is 0, and the problem of aligning supplier’s inventory decision with what is in the best interest of the supply chain goes away. Therefore, we explicitly assume that \( z_0'(0) > 0 \), where prime denotes derivative with respect to \( b \). Next, from the first-order optimality condition we obtain:

\[
b_0 = \ln[t(1)]/\ln(\rho_1), \tag{9}
\]

where, \( z_0(b_0) \geq z_0(b) \) for all \( b \), and for any \( y > 0 \),

\[
t(y) = \frac{-h(1 - \rho_1)}{(h + y\beta)\rho_1 \ln(\rho_1)}. \tag{10}
\]

It is easy to verify that the condition \( z_0'(0) > 0 \) is identical to the inequality \( t(1) < 1 \), which in turn ensures \( b_0 > 0 \). Expression (9) ignores integrality of \( b \) which is often relevant for the industrial application we have in mind. However, since \( z_0 \) is concave in \( b \), we can easily find the integer optimal \( b_0 \) by evaluating \( z_0 \) at the integer floor and the integer ceiling of non-integer \( b_0 \) and setting \( b^*_0 \) equal to that value which maximizes \( z_0 \). The optimum expected profit, which we use as a benchmark in numerical examples, is denoted by \( z^*_0 = z_0(b^*_0) \).
4 The Two-Player Decentralized Model

In this section, we study three different mechanisms governing S-M interactions. We use $z_1$ and $z_2$ to denote the individual expected profit functions of the supplier and the manufacturer respectively, and we use $z_T$ to denote the total expected profit function of the decentralized supply chain, i.e., $z_T = z_1 + z_2 = z_0$. Analytical comparisons are much easier to carry out when target inventory levels are not rounded to integer values. Therefore, all comparisons of algebraic expressions of stock levels are done before rounding. However, we report integer values in numerical examples of Section 5.

4.1 The Fixed-Markup Contract

Since S’s production capacity is assumed fixed, $L_1$ can be affected only through the target inventory level $b$. Suppose M’s purchase managers stipulate that S maintain a target inventory level of $b$ components, which is equivalent to specifying an acceptable lead time distribution $L_1$. In the resulting supply contract, S proposes the corresponding wholesale price $r(b)$. When viewed from the perspective of the supplier, this is equivalent to the scenario in which S offers a menu of $[r(b), b]$ combinations for M to choose from. We further assume that the supplier uses a fixed markup of $\gamma > 0$ above its cost to determine the wholesale price charged to the manufacturer, i.e., $r(b) = [1 + \gamma]c(b)$, where $c(b) = c_1 + \frac{h(b)}{\lambda}$ is the supplier’s per-unit production and inventory cost. Cost plus markup is a commonly used industry practice for setting prices in supply contracts.

The expected profit functions of the individual decision makers and the overall supply chain are $z_1(b, r(b)) = \lambda \gamma c(b)$, $z_2(b, r(b)) = \lambda [\pi(b) - r(b) - c_2]$, and $z_0(b) = \lambda [\pi(b) - c(b) - c_2]$. It is easy to verify that $z_2(b)$ is concave in $b$. The manufacturer will find this type of contract attractive only if there is at least one $b \geq 0$ for which $z_2 \geq 0$. Hereafter, we assume that is the case. We use $b_m$ to denote the optimal stock level for the manufacturer. Then, from the first-order optimality equation, $b_m$ turns out to be as follows:

$$b_m = \max\{0, \frac{\ln\left( \frac{1}{1-\gamma} \right)}{\ln(\rho_1)} \}.$$  \hspace{1cm} (11)

In finding $b_m$, we have used the fact that $z_2$ being concave, its maximum is either the unconstrained maximum, or zero, whichever is greater.

Recall that the stocking level that is optimal for the supply chain is denoted by $b_0$. When $b_m = 0$, it is clearly less than $b_0$ from our assumption in Section 3.1 that $b_0 > 0$. Otherwise, upon comparing (11) to (9), we find that $b_m = b_0$ only when $\gamma = 0$. Furthermore, since $\frac{\partial b_m}{\partial \gamma} < 0$, i.e., $b_m$ is decreasing in $\gamma$, $b_m < b_0$ for all $\gamma > 0$. That is, unless S is willing to supply components
at cost, \( z_1(b_m, \gamma) + z_2(b_m, \gamma) \leq z_0(b_0) \). Since the supplier will not accept/offer a contract in which its contribution margin is zero, the decentralized supply chain’s performance is suboptimal. This phenomenon is also known as double marginalization (see, for example, Spengler 1950).

### 4.2 The Simple Revenue-Sharing Contract

Under a simple revenue-sharing contract, M chooses a value of revenue-fraction and S responds by picking the target inventory level. For a particular combination of the parameters \( \alpha \) and \( b \), the expected profit functions of S and M can be written as follows:

\[
z_1(\alpha, b) = \lambda \alpha (p_0 - \beta \bar{L}) - h \bar{I} - \lambda c_1. \tag{12}
\]

\[
z_2(\alpha, b) = \lambda (1 - \alpha) (p_0 - \beta \bar{L}) - \lambda c_2. \tag{13}
\]

The dependence of \( \bar{L} \) and \( \bar{I} \) on \( b \) is not shown explicitly in above equations for notational simplicity. A combination \((\alpha, b)\) is feasible if the corresponding values of \( z_1 \) and \( z_2 \) are positive. This happens when \( \alpha_{\text{min}} \leq \alpha \leq \alpha_{\text{max}} \) where \( \alpha_{\text{min}} = \frac{h \bar{L} + \lambda c_1}{\lambda \pi (b)} \), and \( \alpha_{\text{max}} = 1 - \frac{c_2}{\pi (b)} \). In practice, each player may require its minimum expected profit to be greater than some strictly positive threshold. The feasibility condition could be modified to accommodate such stipulations.

Upon adding equations (12) and (13), it can be seen that the total supply chain profit does not depend on \( \alpha \). That is, for each fixed \( b \)

\[
z_T(b) = z_1(\alpha, b) + z_2(\alpha, b), \quad \text{for all } \alpha. \tag{14}
\]

We can prove that for each \( \alpha \), \( z_1 \) is concave in \( b \). Thus, the supplier’s expected-profit maximizing value of the target inventory level can be obtained from the first order optimality equation, as shown below.

\[
b(\alpha) = \frac{\ln \left( t(\alpha) \right)}{\ln(\rho_1)}, \tag{15}
\]

where the function \( t(\cdot) \) is defined in equation (10). Upon differentiating the function \( b(\alpha) \) with respect to \( \alpha \), we observe that \( \frac{\partial b(\alpha)}{\partial \alpha} = \left( \frac{1}{t(\alpha) \ln(\rho_1)} \right) \left( \frac{\partial t(\alpha)}{\partial \alpha} \right) < 0 \). The inequality follows from the fact that \( \ln(\rho_1) \leq 0 \) since \( 0 < \rho_1 < 1 \) and \( \frac{\partial t(\alpha)}{\partial \alpha} < 0 \) [the latter can be confirmed from equation (10)].

Knowing that S will behave in this manner, M can determine the value of \( \alpha \) it should offer to maximize its own profit. It can be shown that M’s expected profit is concave in \( \alpha \) if for each \( \alpha \), S chooses \( b(\alpha) \) as the target inventory level. Proofs of concavity are straightforward and omitted in the interest of brevity. The equilibrium \( \alpha \) and \( b \) are denoted with a subscript \( d \), which stands for the decentralized benchmark model. Our efforts yield the following results:

\[
\alpha_d = \frac{h}{\beta} + \sqrt{\frac{h(\beta + h)(1 - \rho_2)}{\beta \ln(\rho_1) [p_0 \lambda (1 - \rho_2) - \beta \rho_2^2]}}, \tag{16}
\]
and \( b_d = b(\alpha_d) \). We denote the corresponding expected profits with the superscript \( d \), i.e., \( z_1^d = z_1(\alpha_d, b_d) \) and \( z_2^d = z_2(\alpha_d, b_d) \). The term within the square root in equation (16) is non-negative since \( \ln(\rho_1) < 0 \) and \( p_0 - \beta \rho_2/(\lambda(1 - \rho_2)) > 0 \). The latter follows from the fact that \( \pi(\infty) > 0 \).

When calculating the equilibrium pair \((\alpha_d, b_d)\) above, we placed no constraints on the willingness of either player to participate. If for the offered \( \alpha \), there is no \( b \geq 0 \) for which the supplier’s expected profit is strictly positive, it will not participate. Similarly, the manufacturer’s expected profit must also be positive at the chosen \((\alpha, b(\alpha))\) values. However, so long as there is at least one set of feasible contract parameters for which \( z_i \)'s are strictly positive, the Stackelberg equilibrium exists (see, for example, Myerson, 1991). The conditions necessary for the equilibrium to be at \((\alpha_d, b_d)\) are \( b_d \geq 0 \) and \( 0 < \alpha_d < 1 \). These requirements are met when

\[
-\frac{\beta h}{(\beta + h)\lambda \ln(\rho_1)} \leq \pi(\infty) \leq -\frac{\beta \rho_2^2(\beta + h) \ln(\rho_1)}{h\lambda(1 - \rho_1)^2}.
\]

We need an upper bound on \( \pi(\infty) \) to ensure that we obtain a non-negative value of \( b_d \) upon solving equation (15). Also, a non-negative value of \( b_d \) implies that \( \alpha_d < 1 \). In the interest of brevity, we do not show the details for the situation in which one of the players’ participation constraint becomes active.

Since \( b(\alpha) \) is increasing in \( \alpha \) (see arguments immediately after equation (15)), it follows that \( b_0 \geq b_d \), with the equality holding only at \( \alpha_d = 1 \). However, \( \alpha_d = 1 \) is not a feasible equilibrium solution. Therefore, \( b_0 > b_d \) and \( z_0(b_0) > z_T(b_d) = z_1(\alpha_d, b_d) + z_2(\alpha_d, b_d) \). Clearly, we need to evaluate other types of contracts that create incentives for M and S to behave in a manner that is overall optimal for the supply chain and at the same time reap benefits that are at least at the level of \( z_i(\alpha_d, b_d) \) for each \( i \). In this sense, we seek alternate contracts that are both Pareto optimal for each player and globally optimal for the supply chain.

Why do we choose this particular simple revenue sharing contract as the benchmark decentralized contract? There are two reasons why this is reasonable. First, the simple revenue sharing contract arises naturally from the notion that supply chain performance can be improved if M and S share the risks and the rewards. In this scheme, both players’ fortunes are indeed affected by the choice of \( b \). Second, in marketing and operations research literature, supplier-retailer interactions are often coordinated through a shared backorder penalty. Note that a simple revenue sharing contract is equivalent to a backorder penalty contract when revenue is linear in lead time. The two players share a fraction of the penalty \( \beta \bar{L}(b) \) according to the chosen \( \alpha \).
4.3 The Two-Part Revenue-Sharing Contract

Consider now the contract in which M offers different revenue-fractions depending on the choice of target inventory level by S. The base-case fraction of revenue offered is \( \alpha = \alpha_d \). This revenue-fraction is available to S when it chooses any stock level \( b \geq 0 \). At the same time, another revenue fraction \( \alpha \geq \alpha \) is also available to the supplier. However, in order to qualify for this level of revenue sharing S must choose target stock level in the range \( [b_0, \infty) \). If S considers choosing \( \alpha_d \) fraction of the revenue, it will pick \( b = b_d \) and its expected profit will be \( z_1^d \). On the other hand, if it considers \( \alpha \), it will pick \( b = b_0 \). This behavior can be explained as follows. For any \( \alpha < 1 \), the supplier’s optimal stock level \( b(\alpha) \) is less than \( b_0 \) (see equation (15)). Since \( z_1(b) \) is concave in \( b \) that means the supplier’s expected profit is decreasing in \( b \) for all \( b \geq b_0 > b(\alpha) \). Hence, the supplier achieves its maximum profit in the range \( [b_0, \infty) \) at \( b = b_0 \). Now the supplier’s expected profit function can be written as:

\[
z_1(b, \alpha, \alpha, b_0) = \begin{cases}
\lambda \alpha \pi(b) - h \bar{I} - \lambda c_1 & \text{if } b \geq b_0, \\
\lambda \alpha \pi(b) - h \bar{I} - \lambda c_1 & \text{otherwise}.
\end{cases}
\]

The maximum expected profit is the larger of \( z_1^d \) and \( z_1^d = z_1(b_0, \alpha, \alpha, b_0) \). Similarly, the manufacturer’s maximum expected profit is either \( z_2^d \) or \( z_2^d = z_0(b_0) - z_1^d(b_0) \) depending on S’s choice. Both players prefer the option that gives S the higher revenue fraction \( \alpha \) so long as \( z_1^\alpha \geq z_1^d \).

We know from arguments presented earlier that \( z_1(\alpha, b) + z_2(\alpha, b) = z_0(b) \) for every \( b \), and \( z_0(b_0) > z_1^d + z_2^d \). Therefore, it is clear that both players should prefer \( z_1^\alpha \) so long as the two players can distribute the excess profits, \( z_0(b_0) - (z_1^d + z_2^d) \), in such a way as to ensure \( z_1^\alpha \geq z_1^d \). We will next show that \( z_1^\alpha(b_0) \geq z_1^d \) is possible whenever \( \alpha \) is chosen to be in the range \( \alpha_S(b_0) < \alpha < \alpha_M(b_0) \), where \( \alpha_S(b_0) \) and \( \alpha_M(b_0) \) are, respectively, the minimum and maximum values of \( \alpha \). The actual choice of \( \alpha \) will depend on the relative bargaining strength of each party.

Using the fact that \( z_0 \) is concave in \( b \), and that \( z_0(b_0) > z_1^d + z_2^d \) whenever \( \alpha < 1 \), we infer that the function \( z_0 \) intersects a horizontal line drawn at \( z_1^d + z_2^d \) exactly twice (see Figure 2). One of these points is \( b_d \), which is less than \( b_0 \). The other we denote as \( b_u \) and it must be such that \( b_u > b_0 \). In short, the region in which we search for alternate revenue sharing schemes contains \( b_0 \). Starting with \( b = b_d \), and considering unit increments until \( b_u \), we find the minimum and maximum values of \( \alpha \), for each \( b \), for which \( z_1^\alpha \geq z_1^d \). The minimum value is the smallest acceptable to S and the maximum is the most that M will be willing to give up for having a higher target inventory level of components. Let these values, which define iso-profit contours, be denoted \( \alpha_S(b) \) and \( \alpha_M(b) \) respectively. Then, \( \alpha_S(b) = \frac{z_1^d + hI(b) + \lambda \bar{c}_1}{\lambda \pi(b)} \), and \( \alpha_M(b) = 1 - \frac{z_1^d + \lambda c_2}{\lambda \pi(b)} \). Let \( \alpha_u = \alpha_S(b_u) = \alpha_M(b_u) \). At \((\alpha_u, b_u)\), and at \((\alpha_d, b_d)\), we also have \( z_i^\alpha = z_i^d \) for \( i = 1, 2 \). The region containing pairs of \( b \) and \( \alpha \)
in which supply chain performs better than the Stackelberg equilibrium point is the region denoted by ACBDA in Figure 3.

As we stated in the Introduction section, the contract described above is an instance of contracts involving a transfer payment from M to S with two parts. We show this equivalence through formal arguments next. Let $T(b, \alpha, b_o)$ denote the transfer payment from M to S. Then, the expected profit function in (18) can be alternatively written as $z_1(b, \alpha, b_o) = \lambda T(b, \alpha, b_o) - h\bar{I} - \lambda c_1$, where $\bar{T}(b, \alpha, b_o) = \bar{\alpha}\pi(b) - (\bar{\alpha} - \alpha)\pi(b)1_{b < b_o}$, and $1_{\{\cdot\}}$ denotes the indicator function which is 1 if the logical term in the curly brackets is true and zero otherwise. That is, the two-part contract is equivalent to an arrangement in which the supplier receives a wholesale price $r(b) = \bar{\alpha}\pi(b)$ for each unit supplied and is charged a nonlinear penalty whenever orders are delayed. The penalty is zero if the supplier fulfills orders such that the average delay does not exceed $\bar{L}_1(b_0)$. Otherwise the penalty is $(\bar{\alpha} - \alpha)\pi(b)$.

Yet another way to write the transfer function is as $\bar{T}(b, \alpha, b_o) = \alpha\pi(b) + (\bar{\alpha} - \alpha)\pi(b)1_{b \geq b_0}$. Here, the supplier receives a price per item of $r(b) = \alpha\pi(b)$ and a base-stock-level-linked nonlinear holding cost subsidy. This subsidy is zero if the supplier’s average order fulfillment delay is more than $\bar{L}_1(b_0)$. Otherwise the subsidy equals $(\bar{\alpha} - \alpha)\pi(b)$. In either case, the manufacturer need only monitor the average delay it experiences in sourcing components from the supplier. There is no need to monitor the supplier’s stock level in order to implement the two-part scheme.

In closing this section, we draw attention to another possibility in which the players share the revenues, net of holding costs. This results in expected profit functions $z_1(\alpha, b) = \alpha(\lambda(p_0 - \beta \bar{L}) - h\bar{I}) - \lambda c_1$, and $z_2(\alpha, b) = (1 - \alpha)(\lambda(p_0 - \beta \bar{L}) - h\bar{I}) - \lambda c_2$, respectively. In this case, S will always pick $b = b_0$ regardless of the level of $\alpha$, and the value of $\alpha$ will serve to distribute the profits between the two players. Clearly, each player must receive sufficient net revenue to cover its marginal production cost. However, here M has no incentive to offer a value of $\alpha$ greater than the minimum necessary for S’s participation. There may be additional technical difficulties associated with the fact that the holding cost depends on who owns the inventory. For those reasons, we do not explore contracts that also share holding costs in this article.

5 Examples

We begin our investigation by exploring the effect of varying $\rho_1$ and $\rho_2$ (while keeping $\lambda$ constant) on the performance of the supply chain under the simple revenue sharing and the two-part schemes. For the latter, we report the minimum and maximum values of $\alpha$ within which a coordinating
contract is possible. We also report the relative supply chain underperformance at the Stackelberg equilibrium point as a percentage of the maximum expected profit achieved under the coordinating contract.

The results are summarized in Tables 1 and 2. In each case $\lambda = 3$, $p_0 = 23$, $\beta = 7$, $c_1 = c_2 = 1.1$ and $h = 0.33$. In Table 1, $\rho_1$ is varied from 0.68 to 0.94, and $\rho_2$ is set at 0.7. In Table 2, $\rho_2$ is varied from 0.5 to 0.82 and $\rho_1$ is kept fixed at 0.9. The upper bound on $\rho_2$ from equation (8) turns out to be 0.8991. This bound is not very tight since $z^*_d$ becomes negative if $\rho_2$ is greater than 0.82.

Integer values of target inventory levels are denoted by $b_0^*$ and $b_d^*$, and the corresponding expected profits are denoted as $z^*_0$ and $z^*_d$. In each case the non-integer value of $b$ is rounded to an integer (and corresponding $\alpha$ found) such that the supplier’s profit is maximized. This is reasonable since the supplier chooses the value of $b$.

For higher values of $\rho_1$ in Table 1 and $\rho_2$ in Table 2, either $z^*_1$ or $z^*_2$ or both become negative.

### Table 1: Optimal solution when $\rho_1$ is varied.

<table>
<thead>
<tr>
<th>No.</th>
<th>$\rho_1$</th>
<th>$b_0^*$</th>
<th>$z^*_0$</th>
<th>$\alpha^*_0$</th>
<th>$b_d^*$</th>
<th>$\alpha^*_d$</th>
<th>$z^*_d$</th>
<th>$100\left(\frac{z^<em>_0 - (z^</em>_1 + z^<em>_2)}{z^</em>_0}\right)$%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.7</td>
<td>8</td>
<td>43.21</td>
<td>0.11</td>
<td>3</td>
<td>1.77</td>
<td>38.21</td>
<td>7.48</td>
</tr>
<tr>
<td>2</td>
<td>0.75</td>
<td>10</td>
<td>42.52</td>
<td>0.13</td>
<td>4</td>
<td>1.83</td>
<td>36.95</td>
<td>8.79</td>
</tr>
<tr>
<td>3</td>
<td>0.8</td>
<td>13</td>
<td>41.48</td>
<td>0.15</td>
<td>6</td>
<td>2.69</td>
<td>35.03</td>
<td>9.07</td>
</tr>
<tr>
<td>4</td>
<td>0.85</td>
<td>19</td>
<td>39.77</td>
<td>0.19</td>
<td>9</td>
<td>2.73</td>
<td>32.62</td>
<td>11.13</td>
</tr>
<tr>
<td>5</td>
<td>0.9</td>
<td>29</td>
<td>36.36</td>
<td>0.25</td>
<td>16</td>
<td>2.91</td>
<td>28.63</td>
<td>13.28</td>
</tr>
<tr>
<td>6</td>
<td>0.91</td>
<td>32</td>
<td>35.22</td>
<td>0.26</td>
<td>18</td>
<td>2.33</td>
<td>27.56</td>
<td>15.13</td>
</tr>
<tr>
<td>7</td>
<td>0.92</td>
<td>37</td>
<td>33.80</td>
<td>0.28</td>
<td>21</td>
<td>2.04</td>
<td>26.26</td>
<td>16.27</td>
</tr>
<tr>
<td>8</td>
<td>0.94</td>
<td>50</td>
<td>29.53</td>
<td>0.33</td>
<td>31</td>
<td>1.26</td>
<td>22.88</td>
<td>18.26</td>
</tr>
</tbody>
</table>

### Table 2: Optimal solution when $\rho_2$ is varied.

<table>
<thead>
<tr>
<th>No.</th>
<th>$\rho_2$</th>
<th>$b_0^*$</th>
<th>$z^*_0$</th>
<th>$\alpha^*_0$</th>
<th>$b_d^*$</th>
<th>$\alpha^*_d$</th>
<th>$z^*_d$</th>
<th>$100\left(\frac{z^<em>_0 - (z^</em>_1 + z^<em>_2)}{z^</em>_0}\right)$%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>29</td>
<td>45.69</td>
<td>0.22</td>
<td>15</td>
<td>3.64</td>
<td>36.20</td>
<td>12.81</td>
</tr>
<tr>
<td>2</td>
<td>0.55</td>
<td>29</td>
<td>44.14</td>
<td>0.23</td>
<td>15</td>
<td>3.33</td>
<td>34.95</td>
<td>13.27</td>
</tr>
<tr>
<td>3</td>
<td>0.7</td>
<td>29</td>
<td>36.36</td>
<td>0.25</td>
<td>16</td>
<td>2.91</td>
<td>28.63</td>
<td>13.28</td>
</tr>
<tr>
<td>4</td>
<td>0.75</td>
<td>29</td>
<td>31.69</td>
<td>0.26</td>
<td>16</td>
<td>1.87</td>
<td>24.99</td>
<td>15.23</td>
</tr>
<tr>
<td>5</td>
<td>0.8</td>
<td>29</td>
<td>24.69</td>
<td>0.28</td>
<td>17</td>
<td>1.22</td>
<td>19.54</td>
<td>15.93</td>
</tr>
<tr>
<td>6</td>
<td>0.82</td>
<td>29</td>
<td>20.80</td>
<td>0.30</td>
<td>17</td>
<td>0.24</td>
<td>16.63</td>
<td>18.91</td>
</tr>
</tbody>
</table>

A noteworthy pattern is that the supplier’s expected profit at Stackelberg equilibrium point, $z^*_1$, initially increases and then declines. While this broader pattern is observed consistently, small
increases in $\rho_1$ can make $z_1^{d*}$ go either up or down due to fact that $b_d^{*}$ must be an integer. S benefits from having a high utilization of its production resources within a range of $\rho_1$ values, whereas $z_2^d$ and $z_0^*$ decline all along. In contrast, the supply chain as a whole, and both S and M lose when $\rho_2$ increases. In all examples, S earns a smaller portion of the total supply chain profits. This would be mitigated in actual contracts by S requiring a minimum level of profit, which is strictly positive, as a condition for accepting the terms of the contract. The patterns predicted by our models do make sense on an intuitive level. Suppliers whose order books are relatively full are in a better position to extract a greater fraction of overall supply chain profits from the direct-to-market manufacturers. Eventually as replenishments of components take even longer time, the entire supply chain’s profits fall sharply and the supplier’s share of these profits is also smaller. Also, the relative benefit of supply chain coordination is greater when both $\rho_i$’s are high.

Notice that with the exception of some anomalies caused by rounding $b_d$ to integer values, $b_d^{*}$ and $\alpha_d^{*}$ increase in $\rho_1$ and $\rho_2$. This can also be shown analytically. In fact, we have studied the comparative statics for this problem and summarized key findings in Table 3. Strictly speaking, these results hold only when the optimal inventory level is not restricted to be an integer quantity.

<table>
<thead>
<tr>
<th>Effect of Increasing</th>
<th>Centralized System</th>
<th>Decentralized System</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_1$</td>
<td>$\uparrow$</td>
<td>$\uparrow$</td>
</tr>
<tr>
<td>$\rho_2$</td>
<td>—</td>
<td>$\uparrow$</td>
</tr>
<tr>
<td>$h_1$</td>
<td>$\downarrow$</td>
<td>$\downarrow$</td>
</tr>
</tbody>
</table>

Table 3: Effect of increasing parameter values. The symbols $\uparrow$ and $\downarrow$ are used to indicate that the corresponding increase or decrease can be shown analytically. Symbols $\uparrow$ and $\downarrow$ are used when the corresponding effect is seen in numerical examples, but we have not been able to prove it through mathematical arguments. When both up and down arrows are used, it means that the effect is not monotonic. Finally, “—” is used when a parameter does not affect the performance measure.

Table 3 shows that while a higher $\rho_1$ increases $b_d$, greater holding cost ($h_1$) causes the stock level to be smaller. However, both can produce an unexpected benefit to the supplier. The supplier’s profit can go up even as the supply chain as whole performs poorly. In contrast, a larger $\rho_2$ always decreases supply chain’s and each player’s expected profits. Similarly, since $h$ and $c_1$ are closely related through the finance charge for inventory investment, a supplier can be better of if it supplies more expensive components provided all other parameters of the problem remain invariant. In fact, when this finance charge is the dominant component of inventory carrying costs, it is customary to set $h = \nu c_1$, where $\nu$ is the finance charge rate. Using this relationship, we found that the optimal expected profit of S increases over a range of values of $c_1$ and then decreases (see Figure 4). Similarly, when $h$ (or equivalently $\nu$) is increased, $z_1^d$ is once again not monotone. However,
under all these cases, the total supply chain profit is declining. The data used to generate Figure 4 are: $\lambda = 3$, $\rho_1 = 0.85$, $\rho_2 = 0.7$, $p_0 = 23$, $\beta = 7$, $c_2 = 0.2$, and $h = 0.3c_1$ where $c_1 \in [0.02, 2]$.

What causes the supplier to earn greater profits even as the supply chain as a whole is worse off? The answer lies in the nonlinear relationship between contract parameters $\alpha_d$, $b_d$ and $h$. Increasing either $c_1$ or $h$ causes $b_d$ to decrease, which in turn lowers per-unit revenue $\pi(\cdot)$. The average inventory $\bar{I}$ also decreases; however this does not necessarily reduce holding costs. The net effect is a lower total profit for the supply chain. Moreover, $M$ needs to increase $\alpha_d$ in order to prevent $b_d$ from dropping even further. It turns out that over some range of parameter values, $S$ can end up gaining more (due to larger $\alpha_d$ and decrease in average inventory) than what it loses (on account of higher production and/or holding costs). Thus, its expected profit may go up over a range of $h$ and $c_1$ values. Note that in all our examples when $z_1^d$ increases, then so does $z_1^a$ under the two-part revenue sharing contract since $z_1^a \geq z_1^d$.

The percent underachievement, or the amount by which the equilibrium solution underperforms the two-part revenue sharing scheme, increases sharply as utilizations (either $\rho_1$, or $\rho_2$, or both) approach their upper limits. This is shown for a representative data set ($\lambda = 3$, $p_0 = 23$, $\beta = 7$, $c_1 = c_2 = 1.1$, $h = 0.3c_1$, and $\rho_1 \in [0.5, 0.97]$) in Figure 5.

6 The Quoted Lead Time and Lost Sales Models

Numerical examples in the previous section reveal that for reasonable parameter values, lack of coordination can cost in the range of 10–15% in expected profits. Successfully achieving coordination can therefore be a significant source of additional profits for direct-to-market manufacturers. However, $M$’s revenue function may not be of the type shown in equation (1). Therefore, we consider two other types of revenue functions that parallel common business practices. In the first instance, $M$’s revenue depends on a critical fractile of the distribution of delay through a quoted lead time. This model is significantly different from the model in sections 4 and 5 in which revenue depended on realized (or average) delay. In the second model, when delays are large (e.g., due to non-availability of components), customers do not place orders and thus the net demand rate is a function of the stocking decision by $S$. This feature makes the model significantly different from previous models where demand rate is independent of the parameter being coordinated. We show in both cases that the two-part revenue sharing scheme is able to coordinate the supply chain. We provide an abbreviated account of the mathematical developments in the interest of brevity. Selected proofs are presented in Appendices A through D.
6.1 The Quoted Lead Time Model

The earned revenue in this case depends on the probability that M is able to supply within the quoted lead time of $\ell$. The probability that the realized lead time exceeds the threshold $\ell$ can be obtained as follows:

$$\text{Prob}\{L > \ell\} = e^{-(\mu_2-\lambda)\ell} + \rho_1 b^1 k_1(\ell),$$

and

$$k_1(\ell) = \begin{cases} \left(1 - \rho_2\right) \left(\frac{e^{-\mu_2 \ell} - e^{-\mu_1 \ell}}{\rho_2 - \rho_1}\right) & \text{if } \rho_1 \neq \rho_2, \\ e^{-\nu \ell} \rho^1 (1 - \rho)(\mu \ell) & \text{if } \rho_1 = \rho_2 = \rho. \end{cases}$$

Similarly, the expected profit functions for the central planner and decentralized models are as follows (the additional $Q$ in the subscript represents the quoted lead time model):

$$z_{0,Q}(b) = \lambda (\pi - (c_1 + c_2)) - h \bar{I}. \quad (21)$$

$$z_{1,Q}(\alpha, b) = \lambda (\alpha \pi - c_1) - h \bar{I}. \quad (22)$$

$$z_{2,Q}(\alpha, b) = \lambda [(1 - \alpha) \pi - c_2]. \quad (23)$$

Note that the relevant per-unit revenue function $\pi(\cdot)$ is given in (2). The average inventory $\bar{I}$ is calculated from the relationship presented in equation (4). It can be shown that the revenue function $\pi$ is increasing concave in $b$. Since the average inventory, $\bar{I}$, is known to be increasing convex in $b$, it follows that

$$\frac{\partial^2 z_{0,Q}}{\partial b^2} = \lambda \left(\frac{\partial^2 \pi}{\partial b^2}\right) - h \left(\frac{\partial^2 \bar{I}}{\partial b^2}\right)$$

is non-positive. That is, $z_{0,Q}$ is concave in $b$. The optimal $b_{0,Q}$ can therefore be found by setting the first partial derivative of $z_{0,Q}$ to zero. This yields:

$$b_{0,Q} = \frac{\ln(t_{Q}(1))}{\ln \rho_1}, \quad (25)$$

where

$$t_{Q}(\alpha) = \frac{h (1 - \rho_1)}{(\lambda \alpha k_1(\ell) (1 - \rho_1) (r_2 - r_1) - h) \rho_1 \ln \rho_1}. \quad (26)$$

Clearly, for $t_{Q}(1) < 1$, $b_{0,Q} > 0$. Since the problem of coordination goes away when $b_{0,Q} = 0$, we assume $t_{Q}(1) < 1$. This parallels a similar assumption on $t(y)$ in section 3.1.

Turning now to the decentralized version of the quoted lead time model, it can be shown that for each $\alpha$ offered by M, the supplier’s objective function is concave in $b$. Therefore, $b_{d,Q}(\alpha)$, the optimal inventory level given $\alpha$, can be obtained by equating the first derivative of $z_{1,Q}$ to zero. That yields:

$$b_{d,Q}(\alpha) = \frac{\ln(t_{Q}(\alpha))}{\ln \rho_1}. \quad (27)$$
If S picks a target inventory level that maximizes its individual expected profit, then \( z_{2,Q} \), the expected profit function of M, can be shown to be concave in \( \alpha \). The optimal fraction of revenue that M should share with S is thus obtained by setting the first derivative of \( z_{2,Q} \) to zero given that \( b_{d,Q} \) is chosen according to equation (27). This gives:

\[
\alpha_{d,Q} = -k_2(\ell) + \sqrt{\frac{-h(k_2(\ell) + 1)}{\lambda k_3(\ell) \ln \rho_1}},
\]

(28)

where \( k_2(\ell) = \frac{h}{\lambda k_1(\ell)(1 - \rho_1)(r_1 - r_2)} \) and \( k_3(\ell) = r_1 - (r_1 - r_2)e^{-(\mu_2 - \lambda)\ell} \) are used for expositional clarity. Note that both \( k_2(\ell) \) and \( k_3(\ell) \) are non-negative functions. The equilibrium solution for M and S is now completely determined as quantities \( \alpha_{d,Q} \) and \( b_{d,Q} (= b_{d,Q}(\alpha_{d,Q})) \). Since \( t_Q(\alpha) \) is a decreasing function of \( \alpha \), it immediately follows that \( b_{d,Q} \leq b_0,Q \), with the inequality being strict for \( \alpha < 1 \). The necessary conditions for this supply contract to be feasible are: \( b_{d,Q} \geq 0 \) and \( 0 < \alpha_{d,Q} < 1 \). These conditions are met when the following inequality is true.

\[
k_2(\ell) + 1 \geq \max\left[\frac{-h}{\lambda k_3(\ell) \ln \rho_1}, \frac{-h k_3(\ell)}{\lambda \ln \rho_1 (\rho_1 k_1(\ell)(r_1 - r_2))^2}\right].
\]

(29)

From relationships (21)–(23), we see that \( z_{0,Q}(b) = z_{1,Q}(\alpha, b) + z_{2,Q}(\alpha, b) \), and therefore \( z_{0,Q}(b_{0,Q}) > z_{1,Q}^d + z_{2,Q}^d \), where \( z_{i,Q}^d = z_{i,Q}(\alpha_{d,Q}, b_{d,Q}) \) is the equilibrium expected profit of player \( i \). Furthermore, since \( z_{0,Q} \) is concave in \( b \), it follows that a horizontal line drawn at \( z_{1,Q}^d + z_{2,Q}^d \) intersects the function \( z_{0,Q}(b) \) at most at two points, one of which is \( b_{d,Q} \) and the other point is denoted as \( b_{u,Q} \), where \( b_{u,Q} \) must be greater than \( b_0,Q \). A two-part revenue sharing scheme specifies a pair of revenue-fractions \( \underline{\alpha} \) and \( \overline{\alpha} \), and a critical target inventory level \( b_{0,Q} \). Arguments similar to those we presented in section 4.3 can be used to show that when S picks \( \overline{\alpha} \) and the corresponding minimum inventory level \( b_{0,Q} \), both S and M are better off as compared to the equilibrium solution. The coordinating supply contract sets \( \alpha = \alpha_{d,Q} \), and chooses \( \overline{\alpha} \) in the range \( \alpha_S(b_{0,Q}) \leq \alpha < \alpha_M(b_{0,Q}) \), where the iso-cost functions \( \alpha_S(b) \) and \( \alpha_M(b) \) are as defined in section 4.3.

### 6.2 The Lost Sales Model

In this case, demand does not materialize if the component kit is not in stock at the moment a demand arrives. The corresponding revenue function is as shown in equation (3). Expressions for the probability that component kits are not out-of-stock, and the average inventory level are derived slightly differently on account of the fact that all demand is not met. Following relationships can be shown to hold (details can be found in Buzacott and Shanthikumar, 1993).

\[
\kappa(b) = \text{Prob}\{ I > 0 \} = \frac{1 - \rho_1 b}{1 - \rho_1 b + 1}.
\]

(30)
\[ I_L(b) = \frac{b + \rho_1 b + 1}{1 - \rho_1 b + 1} - \left( \frac{\rho_1}{1 - \rho_1} \right). \]  

(31)

Therefore, the expected profit functions for the central planner and decentralized models can be written as shown below.

\[ z_{0,L}(b) = \lambda (r - (c_1 + c_2)) \kappa(b) - h I_L(b). \]  

(32)

\[ z_{1,L}(\alpha, b) = \lambda (\alpha r - c_1) \kappa(b) - h I_L(b). \]  

(33)

\[ z_{2,L}(\alpha, b) = \lambda ((1 - \alpha) r - c_2) \kappa(b). \]  

(34)

Note that the additional subscript \( L \) is used to indicate that it is the lost-sales model. Using standard arguments, it is possible to show that \( \kappa(b) \) is increasing and concave in \( b \), and that \( I_L \) is increasing in \( b \).

The optimal inventory level for the central planner model must be a solution to the first order optimality equation obtained by setting \( \frac{\partial z_{0,L}}{\partial b} = 0 \). After some simplification, it can be shown that the condition that \( \frac{\partial z_{0,L}}{\partial b} = 0 \) is equivalent to the following equation:

\[ -\frac{\rho_1 \omega(1, c_1 + c_2)}{e} = -\frac{1}{\rho_1^{b+1}} e^{-\frac{1}{\rho_1^{b+1}}}, \]  

(35)

where the function \( \omega(\alpha, c) \) is defined as follows:

\[ \omega(\alpha, c) = \frac{\lambda (r \alpha - c) (1 - \rho_1)}{h \rho_1}. \]  

(36)

In the remainder of this article, we suppress the arguments of the function \( \omega \) for notational compactness. The values of these argument will be clear from the context. It is easy to recognize that the solution to equation (35) can be obtained by using the \textit{ProductLog} function. The \textit{ProductLog} function solves for \( x \) in an equation of the form \( xe^x = k \), where \( k \) is a constant, i.e., \( x = \text{ProductLog}[k] \) (see Appendix A for details). For each \( -\frac{1}{e} < k < 0 \), there are two possible solutions to the equation \( xe^x = k \). One of these solutions is greater than -1 and the other is less than -1 (Appendix A).

Setting \( k = \frac{\rho_1 \omega}{e} \), and \( x = -\frac{1}{\rho_1^{b+1}} \), we can write:

\[ b_{0,L} = \frac{-\ln \left( -\rho_1 \text{ProductLog}[\frac{-\rho_1 \omega}{e}] \right)}{\ln(\rho_1)}. \]  

(37)

A non-negative value of \( b_{0,L} \) is obtained only when \( -\rho_1 \text{ProductLog}[\frac{-\rho_1 \omega}{e}] \geq 1 \). That is, when \( x = \text{ProductLog}[\frac{-\rho_1 \omega}{e}] < -\frac{1}{\rho_1} \). Since for each \( -\frac{1}{e} < k < 0 \), only one of the two roots of the function \( xe^x = k \) is less than -1, a unique non-negative \( b_{0,L} \) is obtained upon solving the above equation. It is shown in Appendix B that this value is also a maximum. This is done by proving that \( \frac{\partial z_{0,L}}{\partial b} \) is
negative for all \( b \geq b_{0,L} \) and positive for all \( b \leq b_{0,L} \), i.e., \( z_{0,L} \) is an increasing-decreasing function of \( b \). An alternative representation of equation (37), which is more suitable for computational analysis, is given as follows:

\[
b_{0,L} = \left( \frac{1 + \text{ProductLog}\left[-\frac{\rho_1 \omega}{e}\right]}{\ln \rho_1} \right) - (\omega + 1).
\] (38)

Similar arguments also yield the optimal size of components inventory in the decentralized setting. The resulting expression is identical to equation (38) except that the arguments of the function \( \omega \) are different. Specifically, \( c = c_1 \) and the value of \( \alpha \) is supplied by \( M \). In order to establish that an equilibrium solution exists, we first show that for each \( \alpha \), there is a unique \( b_{d,L} \) that maximizes \( z_{1,L} \). Next, we also show that if \( S \) uses optimal buffer size, then \( M \)'s expected profit is concave in \( \alpha \). Formal proofs of these properties can be found in Appendices B and C. The equilibrium solution for \( S \) and \( M \) is denoted by \((b_{d,L}, \alpha_{d,L})\), and the corresponding expected profits by \((z_{1,L}^{d}, z_{2,L}^{d})\). It is, however, not possible to write an explicit expression for \( \alpha_{d,L} \).

In Appendix D, we prove that \( b_{d,L} < b_{0,L} \). That is, a simple revenue sharing scheme lacks the ability to coordinate the channel. However, \( M \) can induce channel-optimal behavior by offering a share \( \pi \) of revenue if \( S \) picks a buffer size \( b \geq b_{0,L} \), and \( \alpha_{d,L} \) otherwise. For this contract to be at least as good as the equilibrium solution, \( \pi \) must lie in the region defined by \( \alpha_{S}(b_{0,L}) < \pi < \alpha_{M}(b_{0,L}) \), where the iso-cost functions \( \alpha_{S} \) and \( \alpha_{M} \) are determined as follows:

\[
\alpha_{S}(b) = \frac{1}{r} \left( c_1 + \frac{z_{1,L}^{d} + h I_L(b)}{\lambda \kappa(b)} \right).
\] (39)

\[
\alpha_{M}(b) = 1 - \frac{1}{r} \left( c_2 + \frac{z_{2,L}^{d}}{\lambda \kappa(b)} \right).
\] (40)

7 Further Generalizations

So far, we have shown that for each of the three different types of revenue functions, a two-part revenue sharing scheme can coordinate the supply chain. These results are based on the assumption that the demand arrival process is Poisson and that processing times at the supplier’s and manufacturer’s production facility are exponential. To what extent are these results generalizable to more general replenishment systems? We explore that question in this section.

In the sequel, we do not make any specific assumptions about the demand process or the processing times. The supplier chooses the target inventory level \( b \) and replenishes stock according to the base-stock policy. We will show that our earlier results hold so long as \( \pi(b) \) is increasing and strictly concave in \( b \), and for each fixed \( b \), the stationary distributions of \( I \) and \( L_i \) exist and have
finite means. The latter is a mild requirement since stationary distributions exist for a large class of systems (see Zipkin, 2002, Chapter 6 for details), so long as $\rho_i < 1$, for each $i$.

Let $N_1$ denote the stationary distribution of the number of items yet to be processed by the supplier’s production facility. Note that the existence of the stationary distribution of $L_1$ implies existence of $N_1$. Then, it is easy to see that $\bar{I}(b) = E(b - N_1)^+$. Since $(\cdot)^+$ is an increasing convex function of its argument and $b - N_1$ is linear and increasing in $b$, the function $(b - N_1)^+$ is also increasing convex; being the composition of an increasing convex function and an increasing linear function. Moreover, since convexity is preserved under expectation, $\bar{I}(b) = E(b - N_1)^+$ is also convex in $b$. Suppose the manufacturer uses a simple revenue sharing contract. Then, the expected profit functions of the supply chain, the supplier, and the manufacturer can be written as follows:

$$z_0(b) = \lambda[\pi(b) - c_1 - c_2] - h\bar{I}(b)$$  \hspace{1cm} (41)

$$z_1(\alpha, b) = \lambda[\alpha\pi(b) - c_1] - h\bar{I}(b)$$  \hspace{1cm} (42)

$$z_2(\alpha, b) = \lambda[(1 - \alpha)\pi(b) - c_2]$$  \hspace{1cm} (43)

Each of the above expressions is concave in $b$. This follows from our assumption that $\pi(b)$ is concave, the fact that $-\bar{I}(b)$ is concave, and the fact that the sum of concave functions is concave.

The supply chain profit is maximized when the target inventory level is set at $b_0$, which satisfies the following optimality equation:

$$b_0 = \{b : \pi'(b) = \left(\frac{h}{\lambda}\right)\bar{I}'(b)\},$$  \hspace{1cm} (44)

where prime denotes derivative with respect to $b$. Similarly, for any $\alpha \in (0, 1)$, the optimal target inventory level for the supplier satisfies the following equation:

$$b(\alpha) = \{b : \pi'(b) = \left(\frac{h}{\alpha\lambda}\right)\bar{I}'(b)\}.$$  \hspace{1cm} (45)

Comparing (44) and (45), we see that so long as $\alpha < 1$, $b_0 \neq b(\alpha)$. In fact since $\pi'(b)$ is decreasing in $b$, and the right-hand side of equation (45) is larger than the right-hand side of (44), it follows that $b(\alpha) < b_0$ for all $0 < \alpha < 1$. This inequality also holds for the equilibrium revenue-fraction $\alpha_d$.

From the definitions of expected profit functions, it is easy to see that for each $b$, $z_0(b) = z_1(\alpha, b) + z_2(\alpha, b)$, irrespective of the value of $\alpha$. Also, since $b_0$ maximizes $z_0$, it follows that $z_0(b_0) > z_0(b_d) = z_1(\alpha_d, b_d) + z_2(\alpha_d, b_d)$, where $(\alpha_d, b_d)$ are the equilibrium contract parameters. That means a superior contract always exists as long as the two players can agree upon how to divide the gain from coordination. The magnitude of the benefit from coordination is $z_0(b_0) - z_0(b_d)$, independent of the revenue-fraction used. It can be shown that the two-part revenue sharing
contract incentivises the supplier to pick stock level $b_0$ and that the resulting contract is Pareto optimal for both the players. These arguments are similar to what we presented in section 4.3. We omit the details in the interest of brevity.

Contracts in retail environment use techniques such as buy back provisions, quantity discounts, and subsidies. The two-part revenue sharing scheme proposed in this article is a nonlinear contract, which is similar in spirit to quantity discounts. Quantity discounts are a popular incentive mechanism used in the marketing literature (see, for example, Dolan, 1987). In a sequel to this article, we have found that a two-part contract also works when the manufacturer’s demand is for more standard products and therefore, it has the option to carry finished goods inventory. These studies help establish the suitability of two-part contracts in several different types of supplier–manufacturer interactions.

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References


Appendix

A  The ProductLog function

The function $\text{ProductLog}[k]$ gives the principle solutions of $x$ in the equation $k = xe^x$. Put differently, if we let $f(x) = xe^x$, then $x = f^{-1}(k) = \text{ProductLog}[k]$. A plot of the function $f(x)$ is shown in Figure A.1. The min $\{f(x)\} = \frac{-1}{e}$, which occurs at $x_0 = -1$. The first derivative of $f(x)$, $\frac{df(x)}{dx}$, equals $e^x(1 + x)$, which is positive if and only if $x \geq -1$. This implies $f(x)$ is a decreasing function over the range $(-\infty, -1]$ and an increasing function when $x \in [-1, \infty)$. For each $-\frac{1}{e} < k < 0$, there are exactly two roots of $f(x) = k$. One of these roots is greater than $-1$ (denoted by $x_2$ in Figure A.1) and the other is less than $-1$ (denoted by $x_1$).

B  The Optimality of $b_{0,L}$

When solving for $b_{0,L}$ from equation (35), it is easy to recognize the following equivalences with the ProductLog function described in Appendix A: $k = -\frac{\rho_1\omega}{e}$, and $x = -\frac{1}{\rho_1+1}$. The value of $x$ is less than $-1$ for $0 < \rho_1 < 1$, and $b \geq 0$. The value of $k$ is between $0$ and $-\frac{1}{e}$ since $\omega > 0$. Therefore, the non-negative $b_{0,L}$ obtained from solving (35) is a unique value. It remains to be shown that this value is also a maximum.

Consider the first derivative of $z_{0,L}$, which can be written as follows:

$$\frac{\partial z_{0,L}}{\partial b} = -\left(\frac{h}{(-1 + \rho_1^{1+b})^2}\right) \left(1 + \rho_1^{1+b} \left(-1 + \ln(\rho_1^{1+b+w})\right) \right). \tag{B.1}$$

The RHS of (B.1) is positive (negative) if and only if the term $1 + \rho_1^{1+b} \left(-1 + \ln(\rho_1^{1+b+w})\right)$ is negative (positive). In other words, $\frac{\partial z_{0,L}}{\partial b} \geq 0$ if and only if $\ln(\rho_1^{1+b+w}) \leq 1 - \frac{1}{\rho_1^{1+b+1}}$. The latter simplifies to the inequality:

$$\frac{-1}{\rho_1^{b+1}} e^{-\frac{1}{\rho_1+1}} \leq -\frac{1}{\rho_1}. \tag{B.2}$$
But from (35), we know that \(-[\rho_1^s/c] = -(1/\rho_1)^{b_0+1}e^{-[1/\rho_1^{b_0+1}]}\). It is a straightforward exercise to confirm that the function \(f = (-1/\rho_1^{b+1})e^{-1/\rho_1^{b+1}}\) is increasing in \(b\). That is, condition (B.2) holds for all \(b \leq b_{0,L}\). Similarly, the opposite of (B.2) holds when \(b \geq b_{0,L}\). Thus, we have shown that \(z_{0,L}\) is increasing-decreasing function of \(b\) and therefore achieves its maximum at \(b_{0,L}\).

Proof of optimality of \(b_{d,L}\) is similar since the only difference in expected profit functions of the central planner and the supplier comes from the difference in the arguments of the function \(\omega(c, \alpha)\).

For the central planner model, we have \(c = c_1 + c_2\) and \(\alpha = 1\), whereas for the decentralized regime, \(c = c_1\) and the value of \(\alpha\) is supplied by the M. Since our proof does not depend on the arguments of function \(\omega\), the above result carries over to the decentralized model as well.

C Concavity of \(z_{2,L}\)

We begin by showing below that \(b_{d,L}(\alpha)\) is an increasing and concave function of \(\alpha\).

\[
\frac{\partial b_{d,L}(\alpha)}{\partial \alpha} = -\frac{\partial \omega}{\partial \alpha} \left(\frac{1}{1 + \text{ProductLog}[\frac{\rho_1^s}{e}]}\right) = \frac{\lambda r (\rho_1 - 1)}{h \rho_1} \left(\frac{1}{1 + \text{ProductLog}[\frac{\rho_1^s}{e}]}\right) \geq 0. \tag{C.3}
\]

\[
\frac{\partial^2 b_{d,L}(\alpha)}{\partial \alpha^2} = \left(\frac{\partial \omega}{\partial \alpha}\right)^2 \left(\frac{\ln \rho_1 \text{ProductLog}[\frac{\rho_1^s}{e}]}{1 + \text{ProductLog}[\frac{\rho_1^s}{e}]^3}\right) \leq 0. \tag{C.4}
\]

Similarly,

\[
\frac{\partial \kappa(b)}{\partial \alpha} = \left(\frac{\partial \kappa(b)}{\partial b_{d,L}}\right) \left(\frac{\partial b_{d,L}}{\partial \alpha}\right) \geq 0, \tag{C.5}
\]

and

\[
\frac{\partial^2 \kappa(b)}{\partial \alpha^2} = \left(\frac{\partial \kappa(b)}{\partial b_{d,L}}\right)^2 \left(\frac{\partial^2 b_{d,L}}{\partial \alpha^2}\right) + \left(\frac{\partial^2 \kappa(b)}{\partial b_{d,L}^2}\right) \left(\frac{\partial b_{d,L}}{\partial \alpha}\right)^2 \leq 0. \tag{C.6}
\]

Next, differentiating \(z_{2,L}\) with respect to \(\alpha\), we obtain the following:

\[
\frac{\partial z_{2,L}}{\partial \alpha} = \lambda \left(-r \kappa(b) + ((1 - \alpha)r - c_2) \frac{\partial \kappa(b)}{\partial \alpha}\right). \tag{C.7}
\]

\[
\frac{\partial^2 z_{2,L}}{\partial \alpha^2} = \lambda \left[-2r \frac{\partial \kappa(b)}{\partial \alpha} + ((1 - \alpha)r - c_2) \left(\frac{\partial^2 \kappa(b)}{\partial \alpha^2}\right)\right] \leq 0. \tag{C.8}
\]

Therefore, \(z_{2,L}\) is concave in \(\alpha\), provided that S picks optimal buffer size for each \(\alpha\) picked by the M.

D Proof of \(b_{d,L} < b_{0,L}\)

From equation (C.7), we see that the equilibrium \(\alpha\) satisfies the following property:

\[
\alpha_{d,L} = 1 - \left(\frac{\kappa(b)}{\partial \alpha}\right) + \frac{c_2}{r} < 1 - \frac{c_2}{r}. \tag{D.9}
\]
Furthermore, we show below that $b_{d,L}$ (and $b_{0,L}$) is increasing in $\omega$ which, in turn, is decreasing in $c$.

$$\frac{\partial b_{d,L}}{\partial \omega} = -\frac{1}{1 + \text{ProductLog}[\frac{-\rho_1 e}{\omega}]} \geq 0.$$ (D.10)

$$\frac{\partial \omega}{\partial c} = -\lambda \frac{(1 - \rho_1)}{\rho_1} \leq 0.$$ (D.11)

When put together, the above also imply that $b_{d,L}$ is decreasing in $c$.

$$\frac{\partial b_{d,L}}{\partial c} = \frac{\partial b_{d,L}}{\partial \omega} \times \frac{\partial \omega}{\partial c} \leq 0.$$ (D.12)

Upon comparing $b_{0,L}$ and $b_{d,L}$, we find that the only difference in how they are computed lies in the parameters of $\omega$. $b_{0,L}$ is computed with a higher value of $\alpha (= 1)$, as well as a higher value of $c (= c_1 + c_2)$. Since the optimal buffer size is increasing in $\alpha$, but decreasing in $c$, the relative magnitude of $b_{0,L}$ and $b_{d,L}$ is not immediately obvious. However, it is noted that when all other parameters are kept fixed, $b_{d,L} \geq b_{0,L}$ if and only if $\omega(\alpha_{d,L}, c_1) \geq \omega(1, c_1 + c_2)$. From the definition of $\omega$, this is equivalent to the requirement that $\alpha_{d,L} \geq 1 - \frac{c_2}{r}$. However, from (D.9), it is clear that this condition is never met whenever $b_{d,L} > 0$. 

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Figure 2: $z_0(b) = z_1^0(\alpha, b) + z_2^0(\alpha, b)$ is shown as a function of $b$. Notice that the region in which $z_1^0 + z_2^0 \geq z_1^d + z_2^d$ contains $b_0$. Data: $\lambda = 3$, $\rho_1 = 0.9$, $\rho_2 = 0.7$, $p_0 = 23$, $\beta = 7$, $c_1 = c_2 = 1.1$, and $h = 0.3c_1$. 
Figure 3: Feasible region in $b - \alpha$ space. Point A corresponds to $(\alpha_d, b_d)$ and point B corresponds to $(\alpha_u, b_u)$. The region encompassed by points A and B is the region in which alternate solutions are superior to the Stackelberg equilibrium point. The line C-D corresponds to the feasible value of $\alpha$ for which $b = b_0$. FXE and FYE define regions in which $z_1$ and $z_2$ are both positive. Data: $\lambda = 3$, $\rho_1 = 0.9$, $\rho_2 = 0.7$, $p_0 = 23$, $\beta = 7$, $c_1 = c_2 = 1.1$, and $h = 0.3c_1$. 
Figure 4: Effect of changing cost of components.

Figure 5: Percent under achievement as a function of $\rho_1$. 
Figure A.1: Graph of \( f(x) = x e^x \)