Supply Contracts for Direct-to-Market Manufacturers

ABSTRACT

Direct-to-market manufacturers promise a great deal of product customization fast. Since carrying large amounts of finished goods inventory is not economical, such firms need to create incentives for their suppliers to carry adequate amounts of components inventory upstream in the supply chain from which custom products can be produced quickly on demand. In this paper, we study the effectiveness of revenue-sharing contracts as means by which the direct-to-market manufacturers can coordinate supplier’s stocking decisions. We show that a simple revenue-sharing mechanism does not achieve this goal, and propose a two-part scheme which does. That is, through revenue sharing, the manufacturer can create incentives for the supplier to choose the optimal stocking level contingent upon its own choice of the level of finished goods inventory. However, the manufacturer may not choose the overall optimal finished goods inventory level. The manufacturer’s expected profits may be maximized by acting sub-optimally when it can keep a sufficiently larger share of the smaller overall supply chain profit. Numerical and analytical comparisons are carried out to provide new insights into the behavior of the supplier and the manufacturer. For example, we show that up to a point, a supplier whose production facility is more heavily utilized is more profitable.

Keywords: Supply chain coordination, Incentives, Non-cooperative games, Production economics.
1 Introduction

Many manufacturers make products in advance of demand using a sequence of decisions that proceeds approximately as follows: pick the subset of product configurations to make, forecast demand, choose production quantities, manufacture the chosen items, and market the items to customers. Only a small subset of possible configurations are produced in order to achieve production efficiency. This practice severely limits consumer choices. Furthermore, forecast-based production quantities frequently result in supply and demand mismatches, causing either expensive inventory write-offs or shortages.

An alternative to this approach, that has drawn attention in recent years, is direct-to-market manufacturing. Such manufacturing firms first learn what their customers want, and then quickly make/assemble the desired configuration from modular components and sub-assemblies. Customers get what they want and the manufacturer avoids the cost of shortages and overages (Serwer, 2002). Sharp drop in the cost of acquiring customer preferences, e.g., via the Internet, is one of the leading reasons for this change. Other important reasons that are essential for the financial success of direct-to-market firms include their ability to realize both quick delivery and low-cost manufacturing. These features are made possible by product designs that permit different configurations to be made by mixing and matching modular components and by supply contracts that ensure abundant availability of components. A well known and successful example of such manufacturing operations is the Dell Computer Corporation. It relies on quick deliveries of components from suppliers to keep order turnaround times low; reported to average 18 days (Anonymous, 2000).

Many direct-to-market manufacturers use a uniform promised delivery time as a competitive distinction. If a particular order is delayed beyond the promised time, the manufacturer then expedites the order (possibly using premium freight) and/or offers a discount to customers who accept late deliveries. In either situation, the manufacturer earns a lower revenue per unit. We assume in this article that the manufacturer’s revenue from the sale of an item is a step function. Full revenue is earned only when the items are delivered within the promised delivery time.

Clearly, the offered price, product quality, size of product menu, and promised delivery time are all strategic choices that determine the manufacturer’s relative position in the market (viz. a viz. competitors) and consequently its demand rate. Our focus in this article is on operational issues. Put differently, price, quality, product choices, and promised delivery times are not decision variables. Instead, we study different types of supply contracts that the manufacturer can use to induce its key component supplier to keep the optimal amount of components inventory. In this way, the manufacturer ensures that its interactions with the supplier support its strategic choices.
There are economic reasons why the supplier should hold components inventory. First, a supplier stores inventory at cost which is smaller than the wholesale price charged to the manufacturer, resulting in lower overall finance charges for inventory investment in the supply chain. Second, suppliers may fill orders from many different users of the same components. Therefore, they need a smaller common stock of components to achieve the same service level. In fact, we have anecdotal evidence that many manufacturers require their suppliers to maintain a minimum level of inventory of ready-to-ship components (often as a condition for winning the supply contract) from which they draw supplies as needed. These inventories are typically maintained at a warehouse in close geographical proximity to the location of the manufacturer’s production facilities. We assume that the supplier attempts to maintain the target stock level by performing item-for-item replenishments, i.e., it uses a base-stock inventory control policy.

In some cases customization takes the form of installing customer-specified options that are easy to plug into a common finished-product platform. When that happens, the manufacturer may maintain some finished-goods inventory. That is the situation we model in this paper. [The situation in which the manufacturer maintains no inventory on account of custom processing of orders has been analyzed in Gupta and Weerawat, 2003.] We assume that the manufacturer also controls inventory via a base-stock control policy. Since the supply systems of both the manufacturer and the supplier are queueing systems, base-stock policies are not necessarily optimal for either decision maker. However, such policies are frequently used for their ease of implementation and because in some closely related systems, they can be shown to be optimal (see, for example, Zipkin 2000).

This article evaluates supply contracts in terms of their ability to achieve supply chain coordination. [Coordination occurs when each decision maker chooses the overall optimal stocking policy.] In order to facilitate comparisons, we first present a solution to the benchmark central-planner model. This is followed by a contract that entails mandatory stock keeping by the supplier. Mandatory stock keeping increases supplier’s costs, which are passed on to the manufacturer in the form of a higher per-unit wholesale price. If the supplier and the manufacturer furthermore agree to a cost-plus-a-fixed-markup scheme for determining the wholesale price of components, it can be shown that the concomitant double marginalization leads to suboptimal decisions. That is, the manufacturer will require the supplier to maintain less than the overall optimal amount of inventory. Having thus identified a need for alternate contracting mechanisms, the remainder of the article is devoted to studying revenue-sharing contracts. The strengths and weaknesses of revenue-sharing contracts have been reported in a recent article by Cachon and Lariviere (2000) and we discuss that paper and other related literature in Section 2.
We propose a mechanism of revenue sharing in which the interaction between the supplier and the manufacturer is viewed as a *noncooperative sequential game* with the manufacturer the Stackelberg leader, and the supplier the follower. The manufacturer’s strategy consists of choosing its target inventory level, $s_2$, and a value of revenue-fraction, $\alpha$, that it is willing to share with the supplier. Subsequently, the supplier chooses the target level of components inventory, $s_1$. The probability of the delivery time exceeding the promised lead time is completely determined by the target inventory levels of the components and the finished items. The choice of the inventory levels is thus equivalent to choosing a delivery time distribution. Each combination of $[(s_2, \alpha), s_1]$ is called a *supply contract*. We assume that both players know each other’s costs and revenue parameters. That is, for given terms of the supply contract, the manufacturer and the supplier can calculate each other’s expected payoff. Clearly, only those combinations for which expected payoffs exceed the respective minimums required by the two players to participate constitute the feasible set of strategies.

In Stackelberg games, the leader (manufacturer) chooses a strategy to maximize its own profit from the set of strategies that are feasible for both players. The follower (supplier) picks a contingent stocking policy that maximizes its profit. Thus, so long as the feasible set is non-empty, there exists an equilibrium contract. We show how to obtain the parameters of the equilibrium contract.

For each $s_2$, the supply-chain-optimal components stock level is denoted by $s^o_1(s_2)$. The overall optimal stock levels are denoted by $s^o_1$ and $s^o_2$. We show that the equilibrium solution described above does not guarantee supply chain coordination. In fact, an even stronger notion of lack of coordination holds. The manufacturer may not the choose $s^o_2$, even when contingent upon its choice of $s_2$, the supplier is incentivised to select $s^o_1(s_2)$. For example, when supplier’s costs and/or minimum profit requirements are large, the manufacturer may prefer $s_2 > s^o_2$. Such a choice allows the manufacturer to offer a smaller revenue-fraction and keep a greater share of the total revenue for itself. That is, even though the total profit is smaller, the absolute value of the manufacturer’s share may be larger. [This is in contrast to several previous studies in which the manufacturer’s (retailer’s) share of the expected profits is maximized by choosing contract parameters that also maximize supply chain’s profits.] We therefore define a different notion of coordination. We say that coordination is achieved when for a fixed $s_2$, the incentive scheme offered to the supplier results in the supplier picking $s^o_1(s_2)$. With this definition in hand, we show that a carefully constructed two-part revenue-sharing mechanism does incentivise the supplier to pick $s^o_1(s_2)$ for each feasible $s_2$. It turns out that there are in fact many such contracts, differing in the share of the excess profits that each player receives.
We present several numerical examples to illustrate how the two-part scheme might be implemented. We find that in most cases, the manufacturer offers a share of the revenue that is just enough to ensure the supplier’s participation. The manufacturer’s choice of equilibrium contract parameters is denoted by \((\alpha^e, s_2^e)\). We show that the corresponding equilibrium stocking level chosen by the supplier, \(s_1^e(s_2^e)\), is typically such that \(s_1^e(s_2^e) \leq s_1^o(s_2^e)\). That is, the supplier usually stocks no more than the supply-chain-optimal amount. However, in some cases \(s_1^e(s_2^e)\) can be greater than \(s_1^o(s_2^e)\). This occurs when the central planner picks \(s_1^o(s_2^e) = 0\) and in the decentralized system, supplier’s minimum profit threshold, \(z_1\), is large. In order to ensure supplier’s participation, the manufacturer picks a large \(\alpha^e\), and due to the nonlinear relationship between \(s_1\) and the lead time, this sometimes causes \(s_1^e > 0\).

Although motivated by the example of direct-to-market manufacturing firms, our models are applicable to all those manufacturers whose revenues are lead time sensitive. The two-part revenue-sharing contract provides a means by which such manufacturers may create incentives for their suppliers to take optimal inventory decisions while keeping a greater share of the supply chain’s profits. Our models provide easy-to-use relationships for first-pass estimation of the parameters of such contracts. The models help to explain why in many manufacturer-supplier contracts, the manufacturers may not choose globally optimal stocking policies, even when the contingent supplier’s decision is globally optimal.

The remainder of this article is organized as follows: Section 2 provides a summary of related literature. This is followed in Section 3 with the description of the mathematical model, notation, and basic performance measures. It also contains the benchmark central-planner model. Three decentralized models are reported in Section 4 dealing with, respectively, the mandatory stock keeping, the simple revenue sharing, and the two-part contracts. Section 5 contains examples and insights. A summary of the key results obtained in this paper can be found in Section 6. Proofs of Theorems and key Propositions are shown in the Appendices.

2 Literature Review

Supply contracts have been studied extensively in recent literature as possible mechanisms for achieving coordination. Issues addressed by this literature range from choosing the type of contract (and specifying parameters of a particular type of contract) to the impact of information asymmetry (Anupindi and Bassok, 1999; Corbett and Tang, 1999; Lariviere, 1999; and Tsay et al., 1999). A recent survey of this literature can be found in Cachon (2002).

Nearly all of the earlier marketing and operations management studies take the view that the
supplier (stage-1 decision maker) needs to create incentives for the retailer (stage-2 decision maker) to maintain the right amount of inventory downstream in the supply chain. In contrast, in our model, the manufacturer (equivalent to the retailer in other models) creates incentives for the supplier to keep the channel-optimal level of components inventory upstream in the supply chain. It uses the flexibility to maintain some expensive finished-goods inventory to achieve more favorable terms in its contract with the supplier. In this sense, our approach represents interactions between the decision makers of a pull system, whereas the earlier studies have focused on push systems. Another distinction between the two types of models has to do with how inventory finance charges are calculated in the central-planner versus the decentralized models. In the push-system models, items stored at the supplier are not distinct from the items stored at the retailer. Inventory finance charges are paid by the stage at which inventory resides. Since the retailer buys inventory at a wholesale price which does not equal the supplier’s cost, the finance charges associated with holding the same amount of inventory at the retailer are different in the decentralized model. In pull-system models, all component inventory resides with the supplier. The finance charges for the same amount of component inventory are identical in the central-planner and the decentralized models.

Operations management literature deals with both one period (newsvendor framework) and multiperiod contracts. Examples of the former kind of models can be found in Cachon and Zipkin (1999) and Lariviere (1999). The multiperiod models are considered in Anupindi and Bassok (1999) and Tsay (1999). Various coordinating mechanisms are evaluated for their ability to achieve supply-chain coordination. These include adjustments in price, returns (buy backs), quantity flexibility, and the use of incentives/penalties. In all cases, the replenishment lead times are assumed independent of congestion (of orders) in the supply chain. Furthermore, situations in which revenues depend on lead times are not considered. By introducing these features, and by modeling pull-type systems, we provide more realistic models for direct-to-market manufacturers.

Articles that are related more closely to our study are Cachon and Zipkin (1999), Cachon and Lariviere (2000), Caldentey and Wein (2000), and Gerchak and Wang (1999). All four consider two-stage serial supply chains, but model different situations. In Cachon and Zipkin’s model, the two players choose their individual inventory levels after adopting a base-stock policy. Each is responsible for its holding costs, but both share a portion of backorder costs. Cachon and Zipkin determine optimal base-stock levels under two different scenarios: the echelon-inventory game and the local-inventory game, depending on whether echelon or local inventory is tracked by the two players. They derive equilibrium and system optimal solutions and study the relative magnitude of stock levels under the two scenarios.
Cachon and Lariviere (2000) discuss the strengths and weaknesses of revenue-sharing contracts. Since we too focus on revenue-sharing contracts, many of their observations also apply to our models. Note, however, that they are concerned with supplier-retailer relationships and on modeling items with a short selling season. They show that revenue-sharing contracts can coordinate supply systems even when demand is price-dependent. We show that simple and two-part revenue-sharing contracts do not necessarily lead to coordination in pull supply chains because the manufacturer has incentives to deviate from the overall optimal behavior.

Caldentey and Wein (2002) model the first stage (supplier’s) operations as an $M/M/1$ queue. The retailer replenishes its stock according to a base-stock policy and pays for inventory. The supplier makes investments in capacity. However, there is no further processing of items at stage-2. In our model, the first stage operates like a $M/M/1$ queue; however, some processing is required at the manufacturer’s facility before the product is ready for the customer. Caldentey and Wein assume that the retailer earns a fixed revenue per unit sold and that backorder allocation fraction is exogenously determined. In contrast, we model the situation in which revenue per unit is lead-time sensitive and coordination is achieved by adjusting the share of the revenue that each player receives. Since longer lead times result in smaller per-unit revenue, in effect the two players share a lateness penalty. The key difference is that the manner in which the lateness penalty is shared by the two players is determined by the outcome of their game. It is not an exogenous parameter as assumed by Caldentey and Wein.

Gerchak and Wang (1999) consider kitting operations in which components from several suppliers are needed to complete an order by the retailer/assembler. Revenue sharing is used to induce system-optimal supplier behavior. They use a newsvendor framework (without capacity constraints) and derive conditions under which one supplier will transfer a portion of its revenues to another, in order to maximize its own profits.

Finally, Gupta and Weerawat (2003) study a closely-related problem in which the manufacturer chooses not to carry inventory since orders cannot be processed before knowing customer choices. Their is a special case (with $s_2 = 0$) of the analysis presented in this article. They show that a carefully constructed two-part revenue-sharing scheme leads to supply-chain coordination.

3 Preliminaries

Consider the two-stage supply chain shown in Figure 1. Demands for finished products arrive one at a time according to a Poisson process. Each customer demand triggers a signal to produce at the supplier’s and the manufacturer’s production facilities and a raw-material kit is instantaneously
released to the supplier’s production system. However, since the manufacturer needs component-kits to produce finished products, a production signal may be backlogged at the manufacturer if components are out of stock. The customer demand is met from manufacturer’s inventory store, if a finished item is available, and backlogged otherwise. Backorders are supplied on a first-come-first-served (FCFS) basis. Each player’s production system is modeled as a single-server queue with exponentially distributed processing times. The assumption of exponentially distributed processing times is reasonable when processing times vary significantly on account of machine failures, setups, and different processing requirements for different end products. We use the following notation for model parameters.

\[ i = \text{Player index: Supplier (} i = 1 \text{) and Manufacturer (} i = 2 \text{).} \]
\[ \lambda = \text{Demand arrival rate.} \]
\[ \mu_i = \text{Average production rate for player-}i \]
\[ \rho_i = \lambda/\mu_i = \text{Player-}i \text{ capacity utilization.} \]
\[ s_i = \text{Size of inventory store (base-stock level) for player-}i \]
\[ L_i(\cdot) = \text{Delivery delay due to player-}i. \quad L = L_1 + L_2. \]
\[ c_i = \text{Unit production cost for player-}i \]
\[ \theta(\cdot) = \text{Revenue from the sale of a unit of finished product.} \]
\[ h_i = \text{Inventory holding cost (sum of warehousing and finance charges) at store } i. \]
\[ N_i(\cdot) = \text{Number of units in player-}i \text{'s production system in steady state.} \]
\[ I_i(\cdot) = \text{Number of units in store-}i \text{ in steady state at an arbitrary observation epoch.} \]
\[ p_i(\cdot) = P(L > \ell), \text{where } \ell \text{ is the quoted delivery time.} \]

The function \( \theta(\cdot) \) is a step function which depends on the promised delivery date, \( \ell \), and the order-processing delay \( L \), i.e.,

\[
\theta(s_1, s_2) = \begin{cases} 
  r_1 & \text{if } L \leq \ell, \\
  r_2 & \text{otherwise}. 
\end{cases}
\]  

The expected revenue \( \bar{\theta} \) can be calculated as follows: \( \bar{\theta}(s_1, s_2) = r_1 - (r_1 - r_2)p_i(s_1, s_2) \). We also define the following constants and functions that are used throughout the article for clarity of exposition.

\[
\nu_i = \mu_i(1 - \rho_i); \quad i = 1, 2. \tag{2}
\]

\[
k_0(x) = \begin{cases} 
  \rho_2 \frac{\rho_2^x - \rho_1^x}{\rho_2 - \rho_1}, & \text{if } \rho_1 \neq \rho_2, \\
  x \rho^x, & \text{if } \rho_1 = \rho_2 = \rho. 
\end{cases} \tag{3}
\]

\[
k_1(x) = \frac{1 - \rho_1^x}{1 - \rho_1} - k_0(x). \tag{4}
\]

\[
k_2(x) = \begin{cases} 
  (1 - \rho_2) \left( \frac{\rho_2 e^{-\nu \ell} - \rho_1 e^{-\nu_1 \ell}}{\rho_2 - \rho_1} \right), & \text{if } \rho_1 \neq \rho_2, \\
  e^{-\nu \ell} \rho^x(1 - \rho)(x + \mu \ell), & \text{if } \rho_1 = \rho_2 = \rho. 
\end{cases} \tag{5}
\]
\[ k_3(x) = h_1 + (1 - \rho_1)(\lambda(r_1 - r_2)k_2(x) - h_2k_1(x)). \]  \hspace{1cm} (6)

\[ k_4(x) = \frac{h_1}{\lambda k_2(x)(1 - \rho_1)(r_1 - r_2)}. \]  \hspace{1cm} (7)

\[ k_5(x) = r_1 - (r_1 - r_2)e^{-\nu_2}\rho_2^s. \]  \hspace{1cm} (8)

\[ k_6(x) = 1 - \frac{h_2k_1(x)}{\lambda k_2(x)(r_1 - r_2)}. \]  \hspace{1cm} (9)

Figure 1: Schematic of the two-player supply chain.

Exact expressions for the performance measures of the two-player supply chain shown in Figure 1 can be obtained only in certain special cases. These cases arise when either \( s_1 = 0 \), or \( s_1 \to \infty \) (see Lee and Zipkin, 1992, and Appendix A for details). The main difficulty is that the process of arrival of component kits to manufacturer’s production facility is not a renewal process; in fact, these times are correlated through their dependence on the number of component orders that are yet to be processed by the supplier’s facility. In light of this difficulty, we present models that utilize an approximation to obtain performance measures.

In Appendix A, we show how to obtain closed-form expressions for the various performance measures of interest after approximating the manufacturer’s production facility as an exogenous supply system with exponentially distributed sojourn times with parameter \( \nu_2 \). Evidence that this approximation is reasonable is provided in Lee and Zipkin (1992). Additional evidence can be found in Buzacott, Price and Shanthikumar (1992). The approximate closed-form expressions for functions \( \bar{I}_2 \) and \( p_\ell \) are as shown below (bar notation denotes averages):

\[ \bar{N}_i = \frac{\rho_i}{1 - \rho_i}, \quad i = 1, 2. \]  \hspace{1cm} (10)

\[ \bar{I}_1(s_1) = s_1 - \frac{\rho_1(1 - \rho_1^{s_1})}{1 - \rho_1}. \]  \hspace{1cm} (11)

\[ \bar{I}_2(s_1, s_2) = s_2 - \frac{\rho_2(1 - \rho_2^{s_2})}{1 - \rho_2} - \rho_1^{s_1+1}k_1(s_2). \]  \hspace{1cm} (12)
\begin{align*}
    p_t(s_1, s_2) &= \rho_2^s e^{-\nu_2 t} + \rho_1^s + 1 k_2(s_2).
\end{align*}

Note that the above expressions are exact when \( s_1 = 0 \). In Appendix A, we argue that \( \bar{I}_1(s_1) \) and \( \bar{I}_2(s_1, s_2) \) are increasing convex functions of \( s_1 \) and \( s_2 \) respectively. Properties of functions \( k_1(s_2), k_2(s_2), p_t(s_1, s_2) \) and \( \bar{I}_2(s_1, s_2) \) are proved in Appendix B.

### 3.1 The Benchmark-Central-Planner Model

Let \( z_o \) denote the expected profit of the supply chain when all production and inventory decisions are made by a single decision maker. The subscript “o” signifies a supply-chain-wide perspective. Then,

\begin{align*}
    z_o(s_1, s_2) &= \lambda \bar{\theta}(s_1, s_2) - h_1(\bar{I}_1(s_1) + \bar{N}_2) - h_2 \bar{I}_2(s_1, s_2) - \lambda(c_1 + c_2) \\
    &= \lambda(r_1 - c_1 - c_2) - h_1(\bar{I}_1(s_1) + \bar{N}_2) - h_2 \bar{I}_2(s_1, s_2) - \lambda(r_1 - r_2)p_t(s_1, s_2)
\end{align*}

(14)

For a fixed \( s_2 \), \( z_o(s_1, s_2) \) can be either decreasing (convex or concave depending on parameter values), or an increasing-decreasing function of \( s_1 \). The shape of \( z_o(s_1, s_2) \) is determined primarily by the magnitude of \( k_3(s_2) \). These observations can be used to find the optimal supplier stocking level. [Proof of Theorem 1 below can be found in Appendix C.]

**Theorem 1** For each fixed \( s_2 \geq 0 \), \( z_o(s_1, s_2) \) has the following properties.

1. If \( k_3(s_2) \leq 0 \), then \( z_o(s_1, s_2) \) is a decreasing convex function of \( s_1 \geq 0 \).
2. If \( 0 < k_3(s_2) \leq \frac{h_1(r_1 - 1)}{\rho_1 \ln \rho_1} \), then \( z_o(s_1, s_2) \) is a decreasing concave function of \( s_1 \geq 0 \).
3. If \( k_3(s_2) > \frac{h_1(r_1 - 1)}{\rho_1 \ln \rho_1} \), then \( z_o(s_1, s_2) \) is an increasing-decreasing (concave) function of \( s_1 \geq 0 \).

The optimal level of stock at stage-1 is

\begin{align*}
    s_1^o(s_2) = \begin{cases} 
        0 & \text{if } k_3(s_2) \leq \frac{h_1(r_1 - 1)}{\rho_1 \ln \rho_1}, \\
        -\ln \left[ \left( \frac{\rho_1 \ln \rho_1}{\rho_1 - 1} \right) \left( \frac{k_3(s_2)}{r_1} \right) \right] & \text{otherwise}.
    \end{cases}
\end{align*}

(15)

Note that in this article, we treat \( s_1 \) and \( s_2 \) as continuous variables for the purpose of finding their optimal values and for comparing different coordination schemes. This avoids distortions that can occur on account of discretization.

The optimal manufacturer’s stock level, \( s_2^o \), can be found by carrying out a bounded search. Intuitively, the claim that the optimal \( s_2^o \) lies in a bounded interval can be understood on the basis that the expected revenue per unit of in-stock item is bounded by \( r_1 \) whereas the holding costs at
both stages grow without bounds in $s_1$ and $s_2$. Formal arguments to prove the Proposition stated below can be found in Appendix C

**Proposition 1** The optimal $s_2$ lies in a bounded region defined by $[0, \bar{s}_2]$, where

$$\bar{s}_2 = \min\{s_2 : s^0_1(s_2) = 0, k_2(s_2) \text{ and } z_o(0, s_2) \text{ are decreasing in } s_2, \text{ and } z_o(0, s_2) < z_o\} \quad (16)$$

In the above Proposition, $z_o > 0$ is the minimum profit requirement for the entire supply chain. [In order to carry out comparisons later with two player models, we will set $z_o$ equal to the sum of the minimum profit requirement at stage-1 (denoted by $z_1$), and stage-2 (denoted by $z_2$).] Note that $\bar{s}_2$ is the smallest value of $s_2$ for which all three conditions in (16) are simultaneously satisfied.

4 Two Player Models

In this Section, two autonomous decision makers, the supplier with profit function $z_1$, and the manufacturer with profit function $z_2$, attempt to maximize their individual expected profits. Both players may require their minimum return to be greater than some strictly positive threshold. We refer to these requirements as participation constraints. Recall from Section 3.1 that the minimum expected profits are denoted respectively as $z_1$ and $z_2$.

We consider three different types of contracts. In the first instance, the supplier is required to keep a certain level of components’ stock. It determines the wholesale price it charges the manufacturer based on a fixed markup over its cost. The second type of contract involves simple revenue sharing. Revenue sharing distributes the lead-time sensitive rewards to both participants, creating incentives for them to take actions that support quick order fulfillment. The third contract we propose is the two-part revenue sharing contract. Contingent upon the manufacturer’s choices, it creates incentives for the supplier to take channel-optimal actions. Decision parameters under the three contracts are differentiated by superscripts: $m$ for mandatory stock-keeping model, $e$ for equilibrium parameters with simple revenue sharing, and $p$ for the two-part mechanism. For example, the two-part revenue fraction offered by the manufacturer is denoted by $\alpha^p$.

We show that in all three cases, the manufacturer may not choose $s^*_2$ even when for any $s_2$, the supplier is incentivised to pick $s^*_1(s_2)$. Therefore, we compare the three different mechanisms in terms of how well each allows the manufacturer to induce the supplier to pick the “right” level of inventory. The right inventory level from the manufacturer’s viewpoint is the level that maximizes the overall expected profit in the supply chain; creating the largest possible excess profit over other contracting schemes. The manufacturer is well positioned to keep a significant portion of these
excess profits. The actual amounts accruing to each player may depend on their relative bargaining strengths.

4.1 The Mandatory Stock-Keeping Contract

The manufacturer chooses $s_2$ and also specifies a required base-stock level $s_1$ from its supplier. By choosing the size of both types of inventories, the manufacturer de facto chooses the lead time distribution. The supplier responds with the corresponding wholesale price $w(s_1)$ which we assume equals its costs, plus a fixed markup $\gamma > 0$. Alternatively, one can understand this transaction as a process in which the manufacturer selects from a menu of lead time distributions (determined by the supplier’s stock level) and the corresponding wholesale price combinations offered by the supplier. There is a different menu for each fixed $s_2$.

The supplier’s per-unit production and inventory holding cost, $c(s_1)$ is given as: $c(s_1) = c_1 + \frac{h_1 I_1(s_1)}{\lambda}$, from where it follows that $w(s_1) = [c_1 + \frac{h_1 I_1(s_1)}{\lambda}] [1 + \gamma]$. The supplier’s expected profit is $z_1(s_1) = \gamma [\lambda c_1 + h_1 I_1(s_1)]$. The manufacturer’s expected profit depends on the price charged by the supplier which in turn depends on the required size of inventory $s_1$. Then, for the given markup $\gamma$, the manufacturer’s expected profit is:

$$z_2(s_1, s_2) = \lambda (r_1 - c_1 - c_2) - \lambda (r_1 - r_2) p_1(s_1, s_2) - h_1 [I_1(s_1) + \bar{N}] - h_2 I_2(s_1, s_2) - \gamma [\lambda c_1 + h_1 I_1(s_1)]$$

Using the relationship in (14), equation (17) can be further simplified to yield:

$$z_2(s_1, s_2) = z_0(s_1, s_2) - \gamma [\lambda c_1 + h_1 I_1(s_1)]$$

Recall from Theorem 1 that $z_0(s_1, s_2)$ is either monotone decreasing (which can be either convex or concave) or increasing-decreasing (concave) in $s_1$. Also, $\bar{I}_1(s_1)$ is increasing convex in $s_1$ which implies that $-\gamma [\lambda c_1 + h_1 \bar{I}_1(s_1)]$ is decreasing concave. Therefore, $z_2$ is either the sum of two concave functions of $s_1$, and hence concave, or the sum of two decreasing functions of $s_1$, and hence decreasing in $s_1$. If $z_2$ is concave, the optimal $s_1$ is obtained from the first-order optimality equation. If either $z_2$ is decreasing in $s_1$, or the first-order optimality equation does not have a positive solution, then optimal $s_1 = 0$. Combining these two cases, we can write the optimal value of $s_1$, denoted by $s_1^m(s_2)$, as follows:

$$s_1^m(s_2) = \begin{cases} 0 & \text{if } \frac{k_3(s_2) + h_1 \gamma}{1+\gamma} \leq \frac{h_1 (\rho_1 - 1)}{\rho_1 \ln \rho_1}, \\ -\ln \left( \frac{\rho_1 \ln \rho_1}{\rho_2 - 1} \frac{k_3(s_2) + h_1 \gamma}{h_1 (1+\gamma)} \right) & \text{if otherwise}. \end{cases}$$

The participation constraints will be met if both $z_1$ and $z_2$ are at least at their respective minimum-threshold levels. Comparing $s_1^m(s_2)$ from equation (19) with $s_1^g(s_2)$ given in equation
(15), we find that \( s^m_1(s_2) = s^o_1(s_2) \) only when \( \gamma = 0 \), or \( s^m_1(s_2) = s^o_1(s_2) = 0 \). Furthermore, since \( \partial s^m_1(s_2)/\partial \gamma < 0 \), i.e., \( s^m_1(s_2) \) is decreasing in \( \gamma \), \( s^m_1(s_2) < s^o_1(s_2) \) for all \( \gamma > 0 \) whenever \( s^m_1(s_2) > 0 \). Since \( \gamma = 0 \) is not a rational choice for the supplier, the decentralized supply chain’s performance will be suboptimal unless the manufacturer carries such a large inventory of finished goods that the supplier need not carry any inventory. In other words, \( z_1(s^m_1, s_2) + z_2(s^m_1, s_2) \leq z_0(s^o_1, s_2) \) with the equality holding only when \( s^m_1(s_2) = s^o_1(s_2) = 0 \). The inferior performance in the uncoordinated supply chain is due to double marginalization (Spengler 1950).

4.2 The Simple Revenue-Sharing Contract

In a simple revenue-sharing contract the revenue from the sale of finished goods is divided between the supplier and the manufacturer according to the fraction \( \alpha \). This creates incentives for the supplier to choose inventory levels that improve lead-time performance.

The manufacturer offers a fraction \( \alpha \) of revenues after revealing the target level of finished goods inventory \( (s_2) \). Knowing the values of these parameters, the supplier chooses \( s_1 \). Since the manufacturer can also compute the supplier’s profit maximizing \( s_1 \), it picks parameter \( \alpha \) to maximize its own profits. The expected-profit functions of the supplier and the manufacturer can be written as follows:

\[
\begin{align*}
z_1(\alpha, s_1, s_2) &= \lambda(\alpha r_1 - c_1) - h_1I_1(s_1) - \lambda \alpha(r_1 - r_2)p_\ell(s_1, s_2) \\
z_2(\alpha, s_1, s_2) &= \lambda[(1 - \alpha)r_1 - c_2] - h_1N_2 - h_2I_2(s_1, s_2) - \lambda(1 - \alpha)(r_1 - r_2)p_\ell(s_1, s_2)
\end{align*}
\]  

(20)  

(21)

It is straightforward to check that \( z_0(s_1, s_2) = z_1(\alpha, s_1, s_2) + z_2(\alpha, s_1, s_2) \) for each \( \alpha \). Our goal in this section is to find the equilibrium values of contract parameters \( \alpha, s_1 \) and \( s_2 \). We develop the necessary analysis in two steps. First, we ignore all participation constraints and only require the contract parameters to be non-negative. Later, we show how the solution obtained in the earlier step can be modified to include the effect of participation constraints.

No Participation Constraints

For a fixed \( s_2 \) and \( \alpha \geq 0 \) specified by the manufacturer, the supplier’s objective function is strictly concave in \( s_1 \). Therefore, \( s_{1\alpha}(\alpha, s_2) \), the optimal target inventory level for the supplier, can be obtained by equating the first derivative of \( z_1 \) to zero.

**Theorem 2** For a given \((\alpha, s_2)\), the optimal \( s_1 \) can be obtained as follows:

\[
s_{1\alpha}(\alpha, s_2) = \max\{0, s_{1d}(\alpha, s_2)\} \quad \text{where} \quad s_{1d}(\alpha, s_2) = \frac{-\ln \left[ \frac{p_1 \ln p_1}{p_1 - 1} \left( \frac{\alpha}{k_2(s_2)} + 1 \right) \right]}{\ln p_1},
\]

(22)

13
Let \(\alpha(s_2) = \max\{\alpha: \alpha \geq 0, \text{and} \, s_{1d}(\alpha, s_2) = 0\}\). That is, \(\alpha(s_2)\) is the largest positive \(\alpha\) for which the supplier chooses not to carry any inventory. Note that \(k_4(s_2)\) is non-negative and that \(s_{1d}(\alpha, s_2)\) is increasing in \(\alpha\), which immediately leads to the following Proposition.

**Proposition 2** \(s_{1\alpha}(\alpha, s_2) = s_{1d}(\alpha, s_2) > 0\) only if \(\alpha > \alpha_0(s_2)\), where \(\alpha_0(s_2) = k_4(s_2)\left(\frac{\rho_1-1}{\rho_1 \ln \rho_1} - 1\right)\).

Whenever the manufacturer offers \(\alpha < \alpha_0(s_2)\), which makes \(s_{1d}(\alpha, s_2) < 0\), it follows from the concavity of \(z_1\) that the supplier’s profit function is monotone decreasing in \(s_1 \geq 0\). In that case, the supplier will pick \(s_1 = 0\) (its least-costly stocking level). Otherwise, the supplier will choose its optimal stock level at \(s_{1d}(\alpha, s_2)\), which is strictly positive. Next, we study the behavior of the supplier’s expected profit as a function of \(\alpha\).

**Theorem 3** Given supplier’s response \(s_1 = s_{1\alpha}(\alpha, s_2)\), the function \(z_1(\alpha, s_{1\alpha}(\alpha, s_2), s_2)\) is increasing in \(\alpha\) for each fixed \(s_2\).

Knowing how the supplier responds helps the manufacturer determine an optimal revenue fraction that maximizes its own profit for each \(s_2\). If \(s_{1\alpha} = 0\), then \(z_2\) is decreasing in \(\alpha\) since the manufacturer retains a smaller fraction of revenue with no change in component stock level. In that case, the manufacturer has incentive to offer the minimum revenue fraction possible. If, however, \(s_{1\alpha} > 0\), then the manufacturer chooses the offered revenue fraction as explained below.

**Theorem 4** For a fixed \(s_2 \geq 0\), if the supplier chooses a strictly positive \(s_{1\alpha}(\alpha, s_2)\), then \(z_2(\alpha, s_{1\alpha}, s_2)\) has the following properties.

i. If \(k_3(s_2) \leq 0\), then \(z_2(\alpha, s_{1\alpha}, s_2)\) is decreasing convex in \(\alpha \geq 0\).

ii. If \(0 < k_3(s_2) \leq -\lambda k_4(s_2) k_5(s_2) \ln \rho_1\), then \(z_2(\alpha, s_{1\alpha}, s_2)\) is a decreasing concave function of \(\alpha\), where \(\alpha \geq 0\).

iii. If \(k_3(s_2) > -\lambda k_4(s_2) k_5(s_2) \ln \rho_1\), then \(z_2(\alpha, s_{1\alpha}, s_2)\) is an increasing-decreasing (concave) function of \(\alpha \geq 0\).

Therefore, the optimal revenue fraction can be calculated as follows:

\[
\alpha_d(s_2) = \begin{cases} 
0 & \text{if } k_3(s_2) \leq -\lambda k_4(s_2) k_5(s_2) \ln \rho_1, \\
-k_4(s_2) + k_4(s_2) \sqrt{\frac{-k_3(s_2)}{\lambda k_4(s_2) k_5(s_2) \ln \rho_1}} & \text{otherwise.}
\end{cases}
\] (23)

Recall that \(k_4(s_2), k_5(s_2)\) are non-negative quantities defined in equations (7) and (8). Knowing that the manufacturer offers \((\alpha_d(s_2), s_2)\) to the supplier, the expression for the optimal supplier’s
stock level \( s_{1a}(\alpha_d(s_2), s_2) \) can be obtained in the following simplified form:

\[
s_{1a}(\alpha_d(s_2), s_2) = \begin{cases} 
0 & \text{if either } \alpha_d(s_2) \leq \alpha_0(s_2), \text{ or } k_3(s_2) \leq \frac{-\lambda k_4(s_2)k_5(s_2)}{\ln p_1} \left( \frac{p_1-1}{p_1} \right)^2, \\
-\ln \left[ \frac{p_1 \ln p_1}{p_1-1} \left( \frac{-k_3(s_2)}{\ln p_1} \right) \right] & \text{otherwise.} 
\end{cases}
\]

Next, we compare the supplier’s optimal stock level, \( s_{1a}(\alpha_d(s_2), s_2) \) in the simple revenue sharing scheme with the optimal benchmark stage-1 stock level, \( s_1^0(s_2) \), for the same \( s_2 \). Theorem 5 lists all the cases that can arise. In order to facilitate this comparison, we recast equation (15) as shown below after using the fact that \( \frac{k_3(s_2)}{k_1} = 1 + \frac{k_6(s_2)}{k_4(s_2)} \):

\[
s_1^0(s_2) = \begin{cases} 
0 & \text{if } k_6(s_2) \leq \alpha_0(s_2), \\
s_{1a}(k_6(s_2), s_2) & \text{otherwise.}
\end{cases}
\]

**Theorem 5** Given the manufacturer’s offer of \( (\alpha_d(s_2), s_2) \) to the supplier, the optimal stock level for the supplier \( s_{1a}(\alpha_d(s_2), s_2) \) has the following magnitude relative to the benchmark \( s_1^0(s_2) \).

A) If \( \alpha_0(s_2) < \alpha_d(s_2) \leq k_6(s_2) \), then \( 0 < s_{1a}(\alpha_d(s_2), s_2) \leq s_1^0(s_2) \).

B) If \( \alpha_d(s_2) \leq \alpha_0(s_2) < k_6(s_2) \), then \( 0 = s_{1a}(\alpha_d(s_2), s_2) < s_1^0(s_2) \).

C) If \( \max\{k_6(s_2), \alpha_d(s_2)\} \leq \alpha_0(s_2) \), then \( 0 = s_{1a}(\alpha_d(s_2), s_2) = s_1^0(s_2) \).

**Accommodating Participation Constraints**

**Proposition 3** The participation constraints (minimum expected profit of \( z_i \) for player-i) are satisfied if and only if the manufacturer offers a revenue fraction \( \alpha \) such that \( \alpha_1(s_1, s_2, z_1) \leq \alpha \leq \alpha_2(s_1, s_2, z_2) \), where

\[
\alpha_1(s_1, s_2, z_1) = \frac{\lambda c_1 + h_1 \tilde{I}_1(s_1) + \tilde{z}_1}{\lambda (r_1 - (r_1 - r_2)p_1(s_1, s_2))} \quad \text{and} \\
\alpha_2(s_1, s_2, z_2) = 1 - \frac{\lambda c_2 + h_1 \bar{N}_2 + h_2 \tilde{I}_2(s_1, s_2) + \bar{z}_2}{\lambda (r_1 - (r_1 - r_2)p_1(s_1, s_2))}.
\]

Proof of the Proposition above is straightforward and entails calculating the minimum and maximum revenue fractions that ensure participation by the two players. Also, from (26) and (27), the relationship between \( \alpha_1(s_1, s_2, z_1) \) and \( \alpha_2(s_1, s_2, z_2) \) can be obtained as follows:

\[
\alpha_2(s_1, s_2, z_1) = \alpha_1(s_1, s_2, z_1) + \frac{z_0(s_1, s_2) - (z_1 + z_2)}{\lambda (r_1 - (r_1 - r_2)p_1(s_1, s_2))}.
\]
This implies that $\alpha_2(s_1, s_2, z_2) \geq \alpha_1(s_1, s_2, z_1)$ only when $z_0(s_1, s_2) \geq z_1 + z_2$. This requirement makes sense since the supply chain’s maximum expected profit level must exceed the sum of minimum profit requirements of the two players. Otherwise, the supply chain is not profitable enough for at least one of the two players to participate in a supply contract.

For each fixed $s_2$, we next modify the key contract parameter $\alpha$ to accommodate participation constraints. The resulting revenue-fraction, $\alpha^e(s_2)$, is called the equilibrium revenue fraction. Note that this completely determines the supplier’s response, denoted by $s^e_1(s_2) = s_{1\alpha}(\alpha^e(s_2), s_2)$. We also define $\alpha_{z_1}(s_2)$, the minimum revenue-fraction that meets supplier’s participation constraint, as follows:

$$\alpha_{z_1}(s_2) = \min\{\alpha : z_1(\alpha, s_{1\alpha}(\alpha, s_2), s_2) \geq z_1\}. \quad (29)$$

Our main result of this Section is summarized in a Theorem below.

**Theorem 6** Given $z_2 \leq \max\{z_2(\alpha_{z_1}, s_{1\alpha}(\alpha_{z_1}, s_2), s_2), z_2(\alpha_d(s_2), s_{1\alpha}(\alpha_d(s_2), s_2), s_2)\}$, the equilibrium revenue fraction is obtained as follows:

1. If $\alpha_d(s_2) \leq \alpha_{z_1}(s_2)$, then $\alpha^e(s_2) = \alpha_{z_1}(s_2)$.

2. Given $\alpha_d(s_2) > \alpha_0(s_2) \geq \alpha_{z_1}(s_2)$, if $z_2(\alpha_{z_1}(s_2), s_{1\alpha}(\alpha_{z_1}(s_2), s_2), s_2) \geq z_2(\alpha_d(s_2), s_{1\alpha}(\alpha_d(s_2), s_2), s_2)$, then $\alpha^e(s_2) = \alpha_{z_1}(s_2)$; else $\alpha^e(s_2) = \alpha_d(s_2)$.

3. If $\alpha_d(s_2) > \alpha_{z_1}(s_2) > \alpha_0(s_2)$, then $\alpha^e(s_2) = \alpha_d(s_2)$.

Whereas a formal proof can be found in Appendix C, informal arguments can help to explain the result in Theorem 6. When $\alpha_d(s_2) \leq \alpha_{z_1}(s_2)$, the manufacturer needs to offer a revenue-fraction greater than $\alpha_d$. For any $\alpha > \alpha_d$, its expected-profit is decreasing in $\alpha$, so it will offer the minimum revenue-fraction at which the supplier’s participation constraint is satisfied. This describes the first case in Theorem 6.

In case ii., since $\alpha_{z_1} \leq \alpha_0$, the supplier will not carry any components stock if it is offered the minimum feasible revenue fraction $\alpha_{z_1}$. Therefore, the manufacturer needs to compare its expected profit for the two options, $\alpha_d$ and $\alpha_{z_1}$, and pick the one that results in the larger expected profit. In case iii., the manufacturer’s objective function is concave in $\alpha$, and $\alpha_d$ satisfies the participation constraint. Since $\alpha_d$ is the unconstrained maximizer of the manufacturer’s expected profit, it represents the manufacturer’s best course of action.

It is quite possible that for certain values of $s_2$, a feasible contract does not exist. In some cases, there may not exist a feasible contract for any $s_2 \geq 0$, which signifies that the interests of the
manufacturer and the supplier are fundamentally at odds and cannot be reconciled via a supply contract.

As the Stackelberg leader, the manufacture searches for its optimal stock level, $s^*_2$, where $s^*_2 = \arg\max_{s_2 \geq 0} z_2(\alpha^e(s_2), s_1^e(s_2), s_2)$. We can prove that a bounded search, which is similar to what has been proposed for the benchmark model, can be used to accomplish this task.

**Proposition 4** The optimal manufacturer stock level, $s^*_2$, lies in a bounded region defined by $[0, s^*_2]$ where

$$s^*_2 = \min \left\{ s_2 : \begin{array}{l} k_2(s_2) \text{ and } z_0(0, s_2) \text{ are decreasing in } s_2; \\ \max \{ \alpha_d(s_2), \alpha_z(s_2) \} \leq \alpha_0(s_2); \text{ and} \\ z_2(\alpha^e(s_2), s_1^e(s_2), s_2) < z_2 \end{array} \right\}. \quad (30)$$

Arguments supporting Proposition 4 are as follows. Recall from Property 2 in Appendix B that $k_2(s_2)$ is decreasing in $s_2$ for large $s_2$. When that happens, it can be shown that $s_{1\alpha}(\alpha_d, s_2)$ goes to zero and that it remains at zero thereafter. Also, from arguments leading up to Theorem 4, it follows that when $s_{1\alpha} = 0$, $z_2$ is decreasing in $\alpha$, and hence the manufacturer has incentive to offer the minimum possible revenue fraction. The minimum feasible revenue fraction is $\alpha^e(s_2) = \alpha_{z_1}(s_2)$, for which the supplier receives $z_1$. That means the manufacturer’s share of expected profits is $z_0(0, s_2) - z_1$, which is decreasing in $s_2$ (since $z_0(0, s_2)$ is decreasing). In this region, we propose to search over $s_2 \leq s^*_2$, i.e., so long as the manufacturer’s expected profit is large enough to satisfy its participation constraint.

### 4.3 The Two-Part Revenue-Sharing Contract

We use $z_1^e(s_2)$ and $z_2^e(s_2)$ to denote the supplier’s and the manufacturer’s expected profits, for each $s_2$, under the equilibrium contract of Section 4.2. In the ensuing discussion, we will look for contract parameters for which player-$i$ earns at least $z_i^e(s_2)$. This is because the simple revenue-sharing contract is also available to both players as a special case of the two-part scheme. Our objective in this Section is to show that when $s_1^e(s_2) \neq s_2^0(s_2)$, there exists an alternate revenue fraction, which we denote by $\alpha^a(s_2)$, that causes the supplier to choose $s_1^0(s_2)$.

Let us first review the cases in which a simple contract achieves coordination. This happens in the following situations:

If $\alpha^e(s_2) = k_0(s_2) > \alpha_0(s_2)$, then $s_1^e(s_2) = s_1^0(s_2) > 0. \quad (31)$

If $\max\{\alpha^e(s_2), k_0(s_2)\} \leq \alpha_0(s_2)$, then $s_1^e(s_2) = s_1^0(s_2) = 0. \quad (32)$

In all other cases, the supplier earns less upon choosing $s_1^e(s_2)$, i.e., $z_1(\alpha^e(s_2), s_1^e(s_2), s_2) < z_1^e(s_2)$. However, the equilibrium profit can be achieved by a supplier who keeps $s_1^0(s_2)$ amount of in-
ventory if it receives a revenue fraction that is large enough to offset its additional inventory holding costs. Put differently, we can find a revenue fraction \( \alpha_1(s_1^0(s_2), s_2, z_1^e(s_2)) \) such that \( z_1(\alpha_1(s_1^0(s_2), s_2, z_1^e(s_2)), s_1^0(s_2), s_2) = z_1^e(s_2) \). [The definition of function \( \alpha_1 \) can be found in Equation (26).] It should be clear that \( \alpha_1(s_1^0(s_2), s_2, z_1^e(s_2)) \) is strictly greater than \( \alpha^e(s_2) \). Since for a fixed \( s_2 \), \( z_o(s_1, s_2) \) achieves its maximum at \( s_1^f(s_2) \), the manufacturer will benefit the most by offering the smallest \( \alpha^a(s_2) \) that results in \( s_1^o(s_2) = s_1^0(s_2) \). In that case, the manufacturer skims off all of the surplus profit. The supplier’s expected profit remains fixed at \( z_1^e(s_2) \), but the manufacturer’s profit increases from \( z_o(s_1^f(s_2), s_2) - z_1^e \) to \( z_o(s_1^f(s_2), s_2) - z_1^e(s_2) \). These ideas are collected in the following Theorem.

**Theorem 7** Let \((\alpha^e(s_2), s_1^e(s_2), s_2)\) be the equilibrium parameters of a supply contract under the simple revenue-sharing scheme for which \( s_1^f(s_2) \neq s_1^o(s_2) \). Let \( \alpha^a(s_2) \) be a revenue fraction that lies in the range \([\alpha_1(s_1^0(s_2), s_2, z_1^e(s_2)), \alpha_2(s_1^0(s_2), s_2, z_2^e(s_2))]\). Then the following two-part revenue-sharing contract achieves coordination:

i. If \( \alpha^e(s_2) > k_0(s_2) \), then the manufacturer should offer

\[
\alpha^p(s_2) = \begin{cases} 
\alpha^a(s_2), & \text{for } s_1 \leq s_1^0(s_2), \\
\alpha^e(s_2), & \text{for } s_1 > s_1^0(s_2).
\end{cases}
\]

(33)

ii. If \( \alpha^e(s_2) < k_0(s_2) \), then the manufacturer should offer

\[
\alpha^p(s_2) = \begin{cases} 
\alpha^e(s_2), & \text{for } s_1 < s_1^0(s_2), \\
\alpha^a(s_2), & \text{for } s_1 \geq s_1^0(s_2).
\end{cases}
\]

(34)

As a practical matter, the manufacturer offers an unconditional revenue fraction \( \alpha^e(s_2) \) regardless of the level of inventory kept by the supplier. In all such cases, we know that the supplier will voluntarily choose \( s_1^f(s_2) \). Furthermore, it also offers a higher revenue fraction, which is available only when the supplier chooses inventory level in the prescribed range. This range depends on the relative magnitudes of \( \alpha^e(s_2), k_0(s_2) \) and \( \alpha_0(s_2) \). It so happens that within the specified range, the supplier’s expected profit is always maximized by selecting \( s_1 = s_1^0(s_2) \). The manufacturer should consider \( \alpha^a(s_2) > \alpha_1(s_1^f(s_2), s_2, z_1^e(s_2)) \) to ensure that the supplier prefers \( \alpha^a(s_2) \) over \( \alpha^e(s_2) \).

5 Examples And Insights

We begin this Section by showing the optimal stock levels under the central-planner model, and the simple revenue-sharing and the two-part contracts, for three examples. For this purpose, we have
identified the best integer-valued stock levels $s_1$ and $s_2$. All examples are based on the base-case data set specified below:

$$\lambda = 0.5, \ c_1 = 1, \ c_2 = 4, \ h_1 = 0.05, \ h_2 = 0.8, \ r_1 = 20, \ r_2 = 10, \ \rho_1 = \rho_2 = 0.7, \ z_1 = z_2 = 1, \ \text{and} \ \ell = 10.$$ 

For each example, we report only the those data parameters that are different from the base-case.

**Example 1 Data:** Same as base case.

**Central-Planner:** $s_2^o = 0, \ s_1^o(s_2^o) = 6, \ z_o = 6.4518.$

**Simple Contract:** $s_2^o = 0, \ s_1^o(s_2^o) = 3, \ \alpha^e = 0.2323, \ z_1^e = 1.5135, \ z_2^e = 4.7788, \ z_0^e = 6.2922.$

**Two-part Contract:** $s_2^o = 0, \ s_1^o(s_2^o) = 6, \ \alpha^o(s_2^o) \in [0.2386, 0.2558], \ \text{max} \ z_2(s_2^o) = 4.9383.$

**Example 2 Data:** $c_2 = 2, \ h_2 = 0.4.$

**Central-Planner:** $s_2^o = 1, \ s_1^o(s_2^o) = 6, \ z_o = 7.5475.$

**Simple Contract:** $s_2^o = 4, \ s_1^o(s_2^o) = 1, \ \alpha^e = 0.1776, \ z_1^e = 1.1770, \ z_2^e = 6.1384, \ z_0^e = 7.3154.$

**Two-part Contract:** $s_2^o = 4, \ s_1^o(s_2^o) = 0, \ \alpha^o(s_2^o) \in [0.1786, 0.1792], \ \text{max} \ z_2(s_2^o) = 6.1442.$

**Example 3 Data:** $\rho_1 = 0.98, \ h_2 = 0.4, \ z_1 = z_2 = 0.$

**Central-Planner:** $s_2^o = 1, \ s_1^o(s_2^o) = 48, \ z_o = 4.4201.$

**Simple Contract:** $s_2^o = 6, \ s_1^o(s_2^o) = 6, \ \alpha^e = 0.0833, \ z_1^e = 0.0025, \ z_2^e = 3.3743, \ z_0^e = 3.3768.$

**Two-part Contract:** $s_2^o = 6, \ s_1^o(s_2^o) = 36, \ \alpha^o(s_2^o) \in [0.1306, 0.1999], \ \text{max} \ z_2(s_2^o) = 3.9246.$

Noteworthy patterns exhibited in the above examples are as follows. The manufacturer extracts a large share of the supply chain profits, both in simple and two-part contracts. The supplier fares better when $z_1$ is large (compare Example 1 and 3). When the manufacturer’s production and holding costs decrease, it is able to work out a more favorable contract and keeps an even greater share of supply chain profits (Example 2). The manufacturer may prefer the supplier to keep significantly less than the optimal amount of components inventory and compensate by keeping more finished goods stock, because by doing so it extracts greater overall profits. This effect is more pronounced in the simple contract (Example 3). It is possible to have $s_1^o(s_2^o) > s_1^o(s_2^o)$ (see Example 2). The manufacturer prefers the smaller stocking level since $I_2$ (and therefore the holding cost) is increasing in $s_1$ (see Property 4 in Appendix B). The manufacturer therefore needs to offer a greater revenue-fraction to the supplier to carry less stock in order that the supplier can earn an amount equal to its earnings under the simple revenue-sharing contract.

In all examples cited here, the search for optimal $s_2$ could be terminated when $s_2$ reached about 15 because the optimal expected profit function of the manufacturer is monotonically decreasing and smaller than $z_2$ for larger values of $s_2$. The examples report the range of $\alpha^o$ values and the
maximum earnings for the manufacturer when it offers the two-part contract. The precise amount of surplus profit earned by manufacturer depends on the chosen value of $\alpha^e$.

In Table 1 below, we show the effect of different levels of the minimum profits required by two players to participate in a supply contract. When the supplier demands a higher $z_1$, $\alpha^e(s_2)$ is higher since $z_1 \geq z_1$ is often the binding constraint. However, the supplier’s profit is frequently not at the minimum threshold ($z^e_1(s_2) \neq z_1$) because we obtain the best integer values of $s_1$ and $s_2$. In those cases where the manufacturer does not change $s_2$ on account of an increase in $z_1$, $s^e_2$ and $\alpha^e$ tend to increase whereas $z^e_2$ decreases. When the manufacturer demands a higher $z_2$, the contract parameters typically do not change. This happens because $z^e_2 \geq z_2$ is usually not the binding constraint in manufacturer-supplier contract negotiations. In those situations where $z^e_2 \geq z_2$ is binding, a larger $z_2$ often causes infeasibility since $z^e_1 \geq z_1$ is frequently a binding constraint.

<table>
<thead>
<tr>
<th>$z_1, z_2$</th>
<th>Simple Contract</th>
<th>Two-part Contract</th>
<th>$\max z_2(s^e_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, 1</td>
<td>$s^e_1(s^e_2)$</td>
<td>$s^e_1(s^e_2)$</td>
<td>$\alpha^e(s^e_2)$</td>
</tr>
<tr>
<td>0, 1</td>
<td>0</td>
<td>2</td>
<td>0.0560</td>
</tr>
<tr>
<td>1, 0</td>
<td>1</td>
<td>2</td>
<td>0.2121</td>
</tr>
<tr>
<td>1, 1</td>
<td>1</td>
<td>2</td>
<td>0.2121</td>
</tr>
<tr>
<td>1, 2</td>
<td>1</td>
<td>2</td>
<td>0.2121</td>
</tr>
<tr>
<td>2, 1</td>
<td>1</td>
<td>3</td>
<td>0.2731</td>
</tr>
</tbody>
</table>

Table 1: Contract parameters for different $z_1$ and $z_2$; base-case data.

Table 2 below reports the results from our next set of experiments in which the workload at each of the two facilities is systematically varied. As before, we use the base-case data set and only deviations from these parameters are noted. Notice that the entire supply chain’s profit decreases when either $\rho_1$ or $\rho_2$ increases. Typically, the supplier is required to hold extra stock to compensate for the higher utilization level, with the result that $\alpha^e$ increases. The manufacturer’s profit is decreasing in both $\rho_1$ and $\rho_2$; however up to a point, the supplier’s expected profit may be larger when either facility is more loaded.

<table>
<thead>
<tr>
<th>$\rho_1, \rho_2$</th>
<th>Simple Contract</th>
<th>Two-part Contract</th>
<th>$\max z_2(s^e_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$s^e_1(s^e_2)$</td>
<td>$s^e_1(s^e_2)$</td>
<td>$\alpha^e(s^e_2)$</td>
</tr>
<tr>
<td>0.7, 0.7</td>
<td>3</td>
<td>1</td>
<td>0.2911</td>
</tr>
<tr>
<td>0.8, 0.7</td>
<td>5</td>
<td>2</td>
<td>0.3203</td>
</tr>
<tr>
<td>0.9, 0.7</td>
<td>9</td>
<td>3</td>
<td>0.3037</td>
</tr>
<tr>
<td>0.7, 0.8</td>
<td>3</td>
<td>2</td>
<td>0.3314</td>
</tr>
<tr>
<td>0.7, 0.9</td>
<td>2</td>
<td>6</td>
<td>0.3326</td>
</tr>
</tbody>
</table>

Table 2: Contract parameters for different utilization levels; base-case data with $z_1 = z_2 = 2$.

We also studied the effect of changing $r_1$, $r_2$ and the quoted lead-time $\ell$, when all other para-
meters remain the same. In practice, if the manufacturer is attempting to change these strategic parameters, its demand rate will also change. However, in the experiments we conducted, the demand rate was kept constant to isolate the impact of these parameters. Actual numerical values are not reported to keep this article brief. We noticed the following patterns. We found that the overall supply chain profit increases when either $r_1$, or $r_2$, or $\ell$ increases. If the manufacturer offers the same $(s_2^2, \alpha^e)$ when $r_1$, $r_2$, or $\ell$ is increased, the supplier’s expected profit increases at a faster pace. In general, however, the manufacturer extracts a greater share of increased profits by lowering its own stock level, or the revenue fraction, as appropriate.

Holding costs being a major determinant of where inventory resides in a supply chain, we next studied the impact of different holding cost rates. The results of our experiments are shown in Table 3. Supplier’s stocking level $s_1^e(s_2)$ and expected profit decreases when $h_1$ is larger. At the same time, the supplier needs greater incentives to hold inventory. We noticed that as $h_1$ increases, the manufacturer initially increases $\alpha^e$, but later responds by raising $s_2^e$ which allows it to lower $\alpha^e$. Likewise, when $h_2$ is large compared with $h_1$, the manufacturer can compensate more cheaply by increasing the incentives for the supplier to hold inventory. Both these patterns demonstrate the effectiveness of the two-parameter contract for the manufacturer.

<table>
<thead>
<tr>
<th>$h_1, h_2$</th>
<th>Simple Contract</th>
<th>Two-part Contract</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1^e(s_2^e)$</td>
<td>$s_2^e$</td>
<td>$\alpha^e(s_2^e)$</td>
</tr>
<tr>
<td>0.01, 0.8</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>0.05, 0.8</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>0.25, 0.8</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0.5, 0.8</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>0.8, 0.8</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>0.05, 0.05</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>0.05, 0.4</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>0.05, 0.8</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>0.05, 1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>0.05, 4</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3: Contract parameters for different holding costs; base-case data with $z_1 = z_2 = 2$.

For many manufacturers and suppliers, the holding cost is primarily composed of the finance charge on investment in inventory. Annual finance charge is a fixed proportion of the cost of inventory and can be different for the two players. We studied the effect of varying $c_i$, the production cost of each player, when the finance charge rate is kept constant. This amounts to fixing the ratio of holding costs to production costs ($h_i / c_i$).

Table 4 shows the impact of increasing $c_i$ with a fixed $h_i / c_i$. Upon increasing $c_1$ in this manner,
\( s_1^2(s_2^2) \) generally decreases. In this sense, its impact is similar to the case when only \( h_1 \) increases. The main difference is that the manufacturer always needs to offer higher \( \alpha^e(s_2^2) \) for the same level of \( s_2^2 \). This happens because the supplier needs to be compensated both for higher holding costs and for higher production costs. Likewise, the effect of increasing \( c_2 \) when \( \frac{h_2}{c_2} \) is fixed, is similar to the impact of increasing \( h_2 \). The main difference is that \( z_2^2 \) is always decreasing in \( c_2 \) including the case when \( s_2^2 \) is set at 0. Furthermore, if \( z_2 \geq z_2^* \) becomes binding, then this typically results in an infeasible problem.

<table>
<thead>
<tr>
<th>( c_1, h_1 )</th>
<th>( c_2, h_2 )</th>
<th>Simple Contract</th>
<th>Two-part Contract</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 0.05</td>
<td>4, 0.8</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>2, 0.1</td>
<td>4, 0.8</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3, 0.15</td>
<td>4, 0.8</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4, 0.2</td>
<td>4, 0.8</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>5, 0.25</td>
<td>4, 0.8</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1, 0.05</td>
<td>1, 0.2</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>1, 0.05</td>
<td>2, 0.4</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>1, 0.05</td>
<td>3, 0.6</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>1, 0.05</td>
<td>4, 0.8</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>1, 0.05</td>
<td>5, 1</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4: Contract parameters for different production costs and fixed finance charge; base-case data.

6 Summary

In this paper, we study the problem of designing supply contracts for direct-to-market and other manufacturers whose revenues are lead-time sensitive. These firms need to create incentives for their suppliers to carry adequate amounts of components inventory upstream in the supply chain in order to support quick order fulfillment. In this aspect, this paper is different from previous studies that focus on the problem faced by a supplier who must create incentives for the downstream retailer to stock adequate amounts of finished goods inventory.

We study the effectiveness of revenue sharing as a mechanism for achieving supply chain coordination. We show that a simple revenue-sharing mechanism does not achieve coordination. We propose a two-part scheme through which the manufacturer can create incentives for the supplier to choose the optimal stocking level contingent upon its choice of the level of finished goods inventory. However, even under these circumstances, the manufacturer may not choose the overall optimal finished goods inventory level. The manufacturer’s expected profits are maximized by act-
ing sub-optimally when it can keep a sufficiently greater share of the smaller overall supply chain profit.

Up to a point, a supplier with higher utilization of its production facilities is able to earn greater expected profit when all other parameters are kept constant. When the optimal policy for a central planner specifies that the supplier-stage should keep large amounts of inventory, the manufacturer in the decentralized supply chain overstocks finished goods. By so doing, it reduces the need for components inventory and the share of revenue it must share with the supplier. This behavior can sometimes create situations in which the manufacturer needs to offer a greater revenue fraction to the supplier to carry smaller amount of components inventory. However, in a typical contract, the opposite is true, i.e., the manufacturer creates incentives for the supplier to carry more inventory by offering a greater share of the revenue fraction. The two-part revenue-sharing mechanism is a robust device through which the manufacturer can create incentives for the supplier to choose the desired stocking level, while keeping a significant share of the overall supply chain profits for itself.

Our results depend on the certain modeling assumptions (e.g., Poisson demand arrivals) and rely on the use of an approximation to obtain the distribution function of delivery delay and the average inventory level \( \bar{I}_2 \). A closer examination, however, reveals that the key analytical results actually depend on the properties of certain functions, that are the building blocks of the supplier’s and the manufacturer’s expected-profit functions. [These functions and their properties can be found in Appendix B.] Therefore, the qualitative insights obtained here should apply to more general supply systems for which the expected-profit functions have properties similar to the properties of \( z_o \), \( z_1 \) and \( z_2 \). Efforts to model coordination of inventory decisions by several suppliers of complementary as well as competing components are currently under way.
References


Appendix

A Performance Metrics

For an exact analysis of the two-player supply chain shown in Figure 1, it is important to obtain the joint equilibrium distribution of occupancy in the production facilities of the supplier and the manufacturer. The major difficulty in evaluating this distribution stems from the fact that there is a positive dependence between the arrivals of component kits to the manufacturer’s facility. Therefore, we use an approximation, developed independently by Lee and Zipkin (1992) and Buzacott, Price and Shanthikumar (1992), that can be used to estimate the performance measures of interest.

Let $B_i$ and $K_i$ denote, respectively, the number of items backordered at facility-$i$ and the number of units facility-$i$ needs to produce to bring its inventory level back to the target level $s_i$. It is easy to see that

$$K_i = s_i - I_i + B_i.$$ \hspace{1cm} (A.1)

Furthermore, the quantity $K_i$ relates $N_i, I_i$ and $B_i$ through the following relationships:

$$I_i = [s_i - K_i]^{+}, \quad B_i = [K_i - s_i]^{+},$$

$$N_1 = K_1, \quad N_2 = K_2 - B_1.$$ \hspace{1cm} (A.2)

Thus, a two-dimensional continuous-time-Markov-Chain model can be formulated in terms of the state vector $K = (K_1, K_2)$. Attempts to solve the system of steady-state balance equations for $K$ results in a boundary-value problem, which can only be solved numerically. It can be observed that when $s_i = 0; i = 1, 2$, the supply system becomes an instance of the Jackson networks, which have product-form solutions. The case where $s_1 = 0, s_2 > 0$ is also not hard to analyze (see Lee and Zipkin, 1992, for details). Also, when $s_1 \rightarrow \infty$, the two production facilities are completely decoupled and behave like independent $M/M/1$ queues resulting in another case that can be solved exactly. Apart from these special cases, it does not seem possible to describe the equilibrium performance without resorting to numerical analysis.

When $s_1 > 0$, Lee and Zipkin propose an approximation in which the supplier’s manufacturing facility operates like an exogenous supply system. This amounts to approximating the point process describing the release of units from the supplier’s production facility to the manufacturer’s production facility by a Poisson process. That is, the distribution of occupancy in the manufacturer’s supply system with $s_1 > 0$ is approximated by the equivalent distribution in a system with $s_1 = 0$. Facility-2 then behaves like a $M/M/1$ queue and hence the (approximate) performance measures can be obtained. Lee and Zipkin (1992) (also Buzacott, Price and Shanthikumar (1992)) report extensive numerical examples and conclude that the approximation works sufficiently well to be used in optimization models.

Under the approximation, the equilibrium $K_i, i = 1, 2$ has a discrete phase-type distribution with parameters $(\pi_i, P_i)$, where $\pi_1 = \rho_1 = P_1, \pi_2 = [\rho_1^{s_i+1}, \rho_2(1-\rho_1^{s_i+1})]$ and $P_2 = \begin{bmatrix} \rho_1 & \rho_2(1-\rho_1) \\ 0 & \rho_2 \end{bmatrix}$, which means that $P[K_i > m] = \pi_i P_i^m e$; $e$ being the column vector of ones. By the spectral decomposition of the matrix $P_2$, we can compute its powers as

$$P_2^m = \begin{bmatrix} 1 & \frac{\nu_1}{\nu_1-\nu_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \rho_1^m & 0 \\ 0 & \rho_2^m \end{bmatrix} \begin{bmatrix} 1 & -\frac{\nu_1}{\nu_1-\nu_2} \\ 0 & 1 \end{bmatrix}.$$
where \( \nu_i = \mu_i(1 - \rho_i), i = 1, 2 \). Then, from (A.1), we can readily derive \( \bar{I}_i \) using

\[
\bar{I}_i = \sum_{r=0}^{s_i} (s_i - r) P[K_i = r].
\]

(A.3)

Noting that \( P[K_i = r] \) is independent of \( s_i \), and viewing the above expression as a linear combination of increasing convex functions, we can say that \( \bar{I}_i \) is increasing convex in \( s_i \). The simplified expression for the average number of units in store-\( i \) is shown in equations (11) and (12). In a similar manner, the customer delivery delay (denoted \( L \)) is distributed as a continuous phase-type distribution with parameters \((\gamma_2 P_2^{s_2}, C_2)\), where \( \gamma_2 = [\rho_1^{s_1}, 1 - \rho_1^{s_1}] \) and \( C_2 = \begin{bmatrix} -\nu_1 & \nu_1 \\ 0 & -\nu_2 \end{bmatrix} \), which implies that \( P[L > \ell] = \gamma_2 P_2^{s_2} e^{C_2 \ell} \). The simplified expression for customer delay distribution is shown in (13).

**B Properties of Basic Functions**

This section identifies certain properties of \( k_1(s_2), k_2(s_2), p_\ell(s_1, s_2) \), and \( I_\ell(s_1, s_2) \) that we later use in the proofs presented in Appendix C. In order to make the exposition clearer, we introduce a new function \( \beta \) which is defined as follows:

\[
\beta(\rho_1, \rho_2) = \begin{cases} 
\frac{-\ln \left( \frac{\rho_1}{\rho_2} \right)}{1 - \left( \frac{\rho_1}{\rho_2} \right)} & \text{if } \rho_1 \neq \rho_2, \\
1 & \text{if } \rho_1 = \rho_2 = \rho.
\end{cases}
\]

(B.1)

We also use the following properties of the \( \beta \) function: if \( \rho_2 < \rho_1 \), then \( \beta(\rho_1, \rho_2) < 1 \); and if \( \rho_1 < \rho_2 \), then \( \beta(\rho_1, \rho_2) > 1 \). The derivation of these properties depends on the fact that for any \( x < 1, x \neq 0 \), the natural log function has the property that \( x < -\ln(1 - x) < \frac{x}{1 - x} \). Consider the situation when \( \rho_2 < \rho_1 \) and set \( x = 1 - \frac{\rho_2}{\rho_1} < 1 \). Then the second part of the above inequality implies that \( -\ln(\rho_2/\rho_1) \leq \frac{1 - (\rho_2/\rho_1)}{1 - (\rho_2/\rho_1)} \), which can be rearranged to the inequality: \( \beta = -\frac{\ln \left( \frac{\rho_1}{\rho_2} \right)}{1 - \left( \frac{\rho_1}{\rho_2} \right)} < 1 \).

Similarly, when \( \rho_1 < \rho_2 \), set \( x = 1 - \frac{\rho_1}{\rho_2} < 1 \) and consider the first part of the inequality above. Then, \( 1 - (\rho_1/\rho_2) < -\ln(\rho_1/\rho_2) \), which can be rearranged to give \( \beta = -\ln(\rho_1/\rho_2)/[1 - (\rho_1/\rho_2)] > 1 \).

**Property 1:** \( k_1(s_2) \) is increasing in \( s_2 \).

**Proof:** Upon differentiating equation (4), we see that

\[
\frac{\partial k_1(s_2)}{\partial s_2} = -\ln \rho_2 k_0(s_2) + \rho_1^{s_2} \left( \frac{-\ln \rho_1}{1 - \rho_1} - \beta(\rho_1, \rho_2) \right).
\]

(B.2)

We will show that the right hand side of equation (B.2) is positive. Notice that its first term is positive since \( -\ln \rho_2 > 0 \) and \( k_0(s_2) > 0 \). We therefore focus on the second term. If \( \rho_2 \leq \rho_1 \) then \( \beta(\rho_1, \rho_2) \leq 1 \) and \( \frac{-\ln \rho_2}{1 - \rho_1} - \beta(\rho_1, \rho_2) \geq \frac{-\ln \rho_1}{1 - \rho_1} - 1 > 0 \). That is, if \( \rho_2 \leq \rho_1 \), then the second term of (B.2) is also positive and therefore \( k_1(s_2) \) is increasing in \( s_2 \). By a similar reasoning, note that if \( \rho_2 > \rho_1 \) then \( \rho_1 < \frac{\rho_2}{\rho_1} < 1, \beta(\rho_1, \rho_2) > 1 \) and \( \frac{-\ln \rho_1}{1 - \rho_1} > \frac{-\ln \left( \frac{\rho_1}{\rho_2} \right)}{1 - \left( \frac{\rho_1}{\rho_2} \right)} = \beta(\rho_1, \rho_2) \). This implies that \( \frac{-\ln \rho_1}{1 - \rho_1} - \beta(\rho_1, \rho_2) > 0 \), and the proof is complete.

#
**Property 2:** \( k_2(s_2) \) is decreasing in \( s_2 \) for large \( s_2 \).

**Proof:** By differentiating equation (5), we obtain

\[
\frac{\partial k_2(s_2)}{\partial s_2} = \ln \rho_2 k_2(s_2) + \rho_1^{s_2} \beta(\rho_1, \rho_2) e^{-\nu_1 t} \left( \frac{1 - \rho_2}{\rho_2} \right). \tag{B.3}
\]

Of the two terms on the right hand side of (B.3), the first term is negative (since \( k_2(s_2) > 0 \) for \( s_2 > 0 \) and \( \ln \rho_2 < 0 \)) and the second term is positive. However, when \( s_2 \to \infty \), the term \( \rho_1^{s_2} \to 0 \). This implies that there must exist a sufficiently large value of \( s_2 \) above which the second term is negligible and the right hand side of (B.3) is negative. That is, \( k_2 \) decreasing in \( s_2 \) for large \( s_2 \). This completes the proof.

**Property 3:** \( p_\ell(s_1, s_2) \) is decreasing convex in \( s_1 \) and decreasing in \( s_2 \).

**Proof:** Using equation (13) to take partial derivatives of \( p_\ell \) with respect to \( s_1 \), we obtain the following inequalities:

\[
\frac{\partial p_\ell}{\partial s_1} = \rho_1^{s_1+1} \ln \rho_1 k_2(s_2) < 0. \tag{B.4}
\]

\[
\frac{\partial^2 p_\ell}{\partial s_1^2} = \rho_1^{s_1+1} (\ln \rho_1)^2 k_2(s_2) > 0. \tag{B.5}
\]

The first inequality holds since \( k_2(s_2) > 0 \) and \( \ln \rho_1 < 0 \). The second inequality is immediate upon observing that \( (\ln \rho_1)^2 > 0 \). This completes the first part of the proof that \( p_\ell \) is decreasing convex in \( s_1 \).

A similar approach is taken to prove the second result. Upon differentiating \( p_\ell \) with respect to \( s_2 \), and after some simplification, the following relationship can be obtained:

\[
\frac{\partial p_\ell}{\partial s_2} = \frac{\partial p_\ell}{\partial s_1} + e^{-\nu_2 \ell} p_2 \rho_2 \left( 1 - \rho_2 \rho_1^{s_1+1} \beta(\rho_1, \rho_2) \right). \tag{B.6}
\]

In light of the result proved earlier, the first term on the right hand side of (B.6) is negative. Therefore, to prove that \( p_\ell \) is decreasing in \( s_2 \), it suffices to show that the quantity \( \rho_2 \left( \frac{\rho_2 \ln \rho_2}{1 - \rho_2} + \rho_1^{s_1+1} \beta(\rho_1, \rho_2) \right) \) \leq 0. We prove that by considering three separate cases:

**Case 1:** \( \rho_2 < \rho_1 \) The desired inequality is obtained by the following sequence of arguments.

\[
\left( \frac{\rho_2 \ln \rho_2}{1 - \rho_2} + \rho_1^{s_1+1} \beta(\rho_1, \rho_2) \right) = \rho_2 \left( \ln \frac{\rho_2}{1 - \rho_2} + \rho_1^{s_1} \frac{\rho_1}{\rho_2} \beta(\rho_1, \rho_2) \right) = \rho_2 \left( \frac{\ln \rho_2}{1 - \rho_2} - \rho_1^{s_1} \frac{\ln \rho_2}{1 - \rho_2} \right) \]

\[
< \rho_2 \left( 1 - \rho_1^{s_1} \frac{\ln \rho_2}{1 - \rho_2} \right) \]

\[
< \rho_2 \left( -(1 - \rho_1^{s_1}) \right) \]

\[
\leq 0 \tag{B.7}
\]

The second equality above is obtained after substituting the value of \( \beta \) from (B.1). The first inequality follows from the fact that \( \rho_2 < \rho_1 \) implies that \( \frac{\ln \rho_2}{1 - \rho_2} < \frac{\ln \rho_2}{1 - \rho_1} \). The second inequality comes from the fact that \( x = \rho_2/\rho_1 < 1 \) and therefore \( \frac{\ln x}{1 - x} > 1 \).
Case 2: $\rho_1 < \rho_2$ The arguments we use in this instance proceed as follows:

\[
\left( \frac{\rho_2 \ln \rho_2}{1 - \rho_2} + \rho_1^{s_1 + 1} \beta(\rho_1, \rho_2) \right) = \rho_2 \left( \frac{\ln \rho_2}{1 - \rho_2} + \rho_1^{s_1} \left( \frac{\rho_1}{\rho_2} \right) \left[ -\ln \left( \frac{\rho_1}{\rho_2} \right) \right] \right)
< \rho_2 \left( \frac{\ln \rho_2}{1 - \rho_2} + \rho_1^{s_1} \right)
< \rho_2 (-1 + \rho_1^{s_1})
\leq 0
\]

The first equality is obtained by substituting for $\beta$ from (B.1); the first inequality follows from that fact that $0 < x = \rho_1/\rho_2 < 1$ and therefore $0 < \frac{x \ln x}{1 - x} < 1$. The second inequality uses the argument that for $x = \rho_2 < 1$, $\frac{\ln x}{1 - x} > 1$.

Case 3: $\rho_1 = \rho_2 = \rho$ In this case can simplify the expression as follows:

\[
\left( \frac{\rho_2 \ln \rho_2}{1 - \rho_2} + \rho_1^{s_1 + 1} \beta(\rho_1, \rho_2) \right) = \rho \left( \frac{\ln \rho}{1 - \rho} + \rho^{s_1} \right)
< \rho (-1 + \rho^{s_1})
\leq 0
\]

The underlying arguments are similar to those used in the previous two cases. Hence proved.

**Property 4:** $I_2(s_1, s_2)$ is increasing and concave in $s_1$.

**Proof:** Upon differentiating equation (12) and simplifying, we obtain the following inequalities:

\[
\frac{\partial I_2(s_1, s_2)}{\partial s_1} = -\rho^{1+s_1} \ln \rho \kappa_1(s_2) > 0
\]

\[
\frac{\partial^2 I_2(s_1, s_2)}{\partial s_1^2} = -\rho^{1+s_1} (\ln \rho)^2 \kappa_1(s_2) < 0
\]

Hence proved.

**C Proof of Theorems and Propositions**

1. **Proof of Theorem 1:**

   Upon differentiating equation (14) with respect to $s_1$, it can be confirmed that, for any fixed $s_2 \geq 0$,

   \[
   \frac{\partial z_0(s_1, s_2)}{\partial s_1} = \begin{cases} 
   -\lambda(r_1 - r_2) \frac{\partial p_1(s_1, s_2)}{\partial s_1} - h_1 \frac{\partial I_1(s_1)}{\partial s_1} & \text{if } s_2 = 0, \\
   -\lambda(r_1 - r_2) \frac{\partial p_1(s_1, s_2)}{\partial s_1} - h_1 \frac{\partial I_1(s_1)}{\partial s_1} - h_2 \frac{\partial I_2(s_1, s_2)}{\partial s_1} & \text{if } s_2 > 0.
   \end{cases}
   \]

   Furthermore, $\frac{\partial I_2}{\partial s_1} = -\left( k_1(s_1) \frac{\partial p_1}{\partial s_1} \right)$ when $s_2 > 0$ and $k_1(s_2) = 0$. Substituting these quantities in the equation above, the right hand side of (C.1) can be simplified as:

   \[
   \frac{\partial z_0(s_1, s_2)}{\partial s_1} = -h_1 \frac{\partial I_1(s_1)}{\partial s_1} + \left( \frac{h_2 k_1(s_2)}{k_2(s_2)} - \lambda(r_1 - r_2) \right) \frac{\partial p_1(s_1, s_2)}{\partial s_1}
   \]
\[
= -h_1 \left( 1 + \frac{\rho_1^{s_1+1} \ln \rho_1}{1 - \rho_1} \right) + \left( \frac{h_2 k_1(s_2)}{k_2(s_2)} - \lambda (r_1 - r_2) \right) \frac{k_2(s_2) \rho_1^{s_1+1} \ln \rho_1}{1 - \rho_1} \\
= -h_1 - k_3(s_2) \frac{\rho_1^{s_1+1} \ln \rho_1}{1 - \rho_1}.
\] (C.2)

Similar reasoning leads to the conclusion that \( \frac{\partial^2 z_0(s_1, s_2)}{\partial s_1^2} = -\rho_1^{s_1+1}(\ln \rho_1)^2 k_3(s_2) \). This implies that \( z_0(s_1, s_2) \) is either concave or convex in \( s_1 \) depending on whether \( k_3(s_2) > 0 \) or \( k_3(s_2) \leq 0 \). The optimal level of stock at stage-1 is:

\[
s_1^0(s_2) = \begin{cases} 
0 & \text{if } k_3(s_2) \leq \frac{h_1(\rho_1-1)}{\rho_1 \ln \rho_1}, \\
-\ln \left( \frac{\rho_1 \ln \rho_1}{\rho_1-1} \right) \left( \frac{k_3(s_2)}{h_1} \right) & \text{otherwise.} 
\end{cases}
\] (C.3)

\#

2. Proof of Proposition 1:
Notice that \( s_1^0(s_2) \) and \( k_3(s_2) \) are decreasing in \( s_2 \) when \( k_2(s_2) \) is decreasing in \( s_2 \). In addition, for large \( s_2 \), \( k_2(s_2) \) is decreasing in \( s_2 \). This implies that for some sufficiently large \( s_2 \), \( s_1^0(s_2) \to 0 \). Now, it can be shown that \( z_0(0, s_2) \) is strictly concave in \( s_2 \) when \( k_2(s_2) \) is decreasing in \( s_2 \). That is, once \( z_0(0, s_2) \) reaches its peak it is strictly decreasing in \( s_2 \) thereafter. We propose an upper bound for the optimal \( s_2 \) to be that point at which \( z_0(0, s_2) \) does not meet the minimum profit requirement for the entire supply chain for the first time.

3. Proof of Theorem 2:
It can be seen that for any fixed \( s_2 \),

\[
\frac{\partial z_1(\alpha, s_1, s_2)}{\partial s_1} = -h_1 \left( 1 + \left( \frac{\rho_1^{s_1+1} \ln \rho_1}{1 - \rho_1} \right) \left( 1 + \frac{\alpha}{k_4(s_2)} \right) \right), \text{ and} 
\] (C.4)

\[
\frac{\partial^2 z_1(\alpha, s_1, s_2)}{\partial s_1^2} = -h_1 \left( \frac{\rho_1^{s_1+1}(\ln \rho_1)^2}{1 - \rho_1} \right) \left( 1 + \frac{\alpha}{k_4(s_2)} \right) < 0.
\] (C.5)

The above inequality follows from the fact that \( k_4(s_2) > 0 \) which implies that \( z_1(\alpha, s_1, s_2) \) is concave in \( s_1 \). The optimal \( s_1 \) in Equation (22) is then obtained by equating the first derivative of \( z_1 \) to zero.

\#

4. Proof of Theorem 3:
If \( \alpha \leq \alpha_0(s_2) \), then \( s_{1\alpha}(\alpha, s_2) = 0 \). The fact that \( z_1(0, s_2) \) is monotone increasing in \( \alpha \) for any \( s_2 \) follows from the fact that \( z_1 \) is monotone increasing in \( \alpha \) when \( s_1 \) and \( s_2 \) are fixed.

If \( \alpha > \alpha_0(s_2) \), then \( s_{1\alpha}(\alpha, s_2) > 0 \). Consider two values of the revenue-fraction: \( \alpha_x(s_2) \) and \( \alpha_y(s_2) \) such that \( \alpha_x(s_2) > \alpha_y(s_2) > \alpha_0(s_2) \). Since for fixed \( s_1 \) and \( s_2 \), \( z_1 \) is increasing in \( \alpha \), that means \( z_1(\alpha_x(s_2), s_1, s_2) < z_1(\alpha_x(s_2), s_1, s_2) \). In particular, the preceding inequality also holds when \( s_1 = s_{1\alpha}(\alpha_y, s_2) \). Furthermore, since \( z_1(\alpha_x(s_2), s_{1\alpha}(\alpha_y, s_2), s_2) < z_1(\alpha_x(s_2), s_{1\alpha}(\alpha_x, s_2), s_2) \)
on account of the fact that \( s_{1\alpha}(\alpha_x, s_2) \) is the optimal stocking level for the offered revenue fraction \( \alpha_x \), we have the following final inequality

\[
z_1(\alpha_y(s_2), s_{1\alpha}(\alpha_y, s_2), s_2) < z_1(\alpha_x(s_2), s_{1\alpha}(\alpha_x, s_2), s_2).
\]  
(C.6)

This proves the claim that \( z_1 \), is increasing in \( \alpha > \alpha_0(s_2) \).

5. Proof of Theorem 4:

Recall from Theorem 2 that the supplier always chooses to hold its stock level at either \( s_{1\alpha}(\alpha, s_2) = 0 \) or \( s_{1\alpha}(\alpha, s_2) = s_{1d}(\alpha, s_2) > 0 \). Upon setting \( p_t(\alpha) := p_t(s_{1d}(\alpha, s_2), s_2) \) and \( z_2(\alpha) := z_2(\alpha, s_{1d}(\alpha, s_2), s_2) \), we obtain:

\[
\frac{\partial z_2(\alpha)}{\partial \alpha} = \begin{cases} 
\lambda((r_1 - r_2)p_t(\alpha)) - \lambda(1 - \alpha)(r_1 - r_2) \frac{\partial p_t(\alpha)}{\partial \alpha} & \text{if } s_2 = 0, \\
\lambda((r_1 - r_2)p_t(\alpha) - r_1) - \lambda(1 - \alpha)(r_1 - r_2) \frac{\partial p_t(\alpha)}{\partial \alpha} - h_2 \frac{\partial z}{\partial \alpha} & \text{if } s_2 > 0.
\end{cases}
\]  
(C.7)

Since \( \frac{\partial z}{\partial \alpha} = -\left(\frac{k_1(s_2)}{k_2(s_2)}\right) \frac{\partial p_t}{\partial \alpha} \) when \( s_2 > 0 \) and \( k_1(s_2) = 0 \), the above relationship can be simplified as follows:

\[
\frac{\partial z_2(\alpha)}{\partial \alpha} = -\lambda r_1 + \lambda(r_1 - r_2)p_t - \frac{\partial p_t}{\partial \alpha} \left(\lambda(1 - \alpha)(r_1 - r_2) - h_2 \frac{k_1(s_2)}{k_2(s_2)}\right)
= -\lambda k_5(s_2) - \lambda(r_1 - r_2)(\alpha + k_4(s_2)) \frac{\partial p_t}{\partial \alpha} - \frac{\partial p_t}{\partial \alpha} \left(\lambda(1 - \alpha)(r_1 - r_2) - h_2 \frac{k_1(s_2)}{k_2(s_2)}\right)
= -\lambda k_5(s_2) - \frac{\partial p_t}{\partial \alpha} \lambda(r_1 - r_2)(k_4(s_2) + k_6(s_2)).
\]  
(C.8)

Based on the fact that \( k_4(s_2) + k_6(s_2) = k_3(s_2) \frac{k_4(s_2)}{k_1(s_2)} \) and \( \left(\frac{\lambda(r_1 - r_2)}{h_1}\right) \frac{\partial p_t}{\partial \alpha} = \frac{1}{\ln \rho_1(\alpha + k_4(s_2))^2} \), equation (C.8) leads to the following expressions:

\[
\frac{\partial z_2(\alpha)}{\partial \alpha} = -\lambda k_5(s_2) - \frac{k_3(s_2)}{k_4(s_2)} \ln \rho_1 \left(\frac{k_4(s_2)}{\alpha + k_4(s_2)}\right)^2, \text{ and}
\]  
(C.9)

\[
\frac{\partial^2 z_2(\alpha)}{\partial \alpha^2} = \frac{2 k_3(s_2) k_4(s_2)}{\ln \rho_1(\alpha + k_4(s_2))^2}.
\]  
(C.10)

This implies that \( z_2(\alpha) \) is a concave function in \( \alpha \) only if \( k_3(s_2) > 0 \). Also, the optimal revenue fraction is:

\[
\alpha_d(s_2) = \begin{cases} 
0 & \text{if } k_3(s_2) \leq -\lambda k_4(s_2) k_5(s_2) \ln \rho_1, \\
-k_4(s_2) + k_4(s_2) \sqrt{\frac{-k_3(s_2)}{\lambda k_4(s_2) k_5(s_2) \ln \rho_1}} & \text{otherwise}.
\end{cases}
\]  
(C.11)

6. Proof of Theorem 5:

Starting with the situation in item A of the Theorem, note that \( \alpha_d(s_2) > \alpha_0(s_2) \) implies that \( s_{1\alpha}(\alpha_d(s_2), s_2) = s_{1d}(\alpha_d(s_2), s_2) > 0 \). Now, from the fact that \( s_{1d} \) is increasing in \( \alpha \) and that \( \alpha_d(s_2) < k_6(s_2) \), it is easy to see that \( s_{1d}(\alpha_d(s_2), s_2) < s_{1d}(s_0(s_2), s_2) = s_0'(s_2) \). This completes the proof of item A of Theorem 5.
If \( \alpha_d(s_2) \leq \alpha_0(s_2) \) (as in Cases B and C), then there are two possibilities: either \( k_0(s_2) > \alpha_0(s_2) \) (which means \( s_1^o(s_2) > 0 \)) or \( k_0(s_2) \leq \alpha_0(s_2) \) (which implies \( s_1^o(s_2) = 0 \)). In both these cases \( s_{1\alpha}(\alpha_d(s_2), s_2) = 0 \). These observations lead to the two cases described in items B and C.

Among all the different ways of ordering quantities \( \alpha_0(s_2) \), \( \alpha_d(s_2) \) and \( k_0(s_2) \), one case is not considered in Theorem 5. The situation arises when \( \alpha_0(s_2) < k_0(s_2) < \alpha_d(s_2) \). It can be shown that this ordering is not feasible. The proof is by contradiction and proceeds as follows.

1. Upon substituting \( \alpha = k_0(s_2) \) in Equation (22), and from the identity in Equation (25), we see that \( s_1^o(s_2) = s_{1d}(k_0(s_2), s_2) \).
2. For each fixed \( s_2 \), the following inequality holds as a consequence of the definition of \( s_1^o(s_2) \):
\[
z_o(s_1, s_2) \leq z_o(s_{1d}(k_0(s_2), s_2), s_2) \quad \forall s_1 \geq 0.
\]
3. Since \( z_o(s_1, s_2) = z_1(\alpha, s_1, s_2) + z_2(\alpha, s_1, s_2) \) for each \( \alpha \), we see that
\[
z_1(\alpha, s_1, s_2) + z_2(\alpha, s_1, s_2) \leq z_o(s_{1d}(k_0(s_2), s_2), s_2) = z_1(k_0(s_2), s_1(k_0(s_2), s_2), s_2) + z_2(k_0(s_2), s_1(k_0(s_2), s_2), s_2).
\]
In particular, this inequality holds when on the left hand side, we substitute \( \alpha = \alpha_d(s_2) \).
4. Since \( \alpha_d \) is the optimal revenue-fraction in the decentralized model, we know that
\[
z_1(\alpha_d(s_2), s_1(\alpha_d(s_2)), s_2) \geq z_1(k_0(s_2), s_1(k_0(s_2), s_2), s_2), \quad \text{and that}
\]
\[
z_2(\alpha_d(s_2), s_1(\alpha_d(s_2)), s_2) \geq z_2(k_0(s_2), s_1(k_0(s_2), s_2), s_2).
\]
Items 3 and 4 contain a contradiction since item 4 shows that there exists another value of \( \alpha \) at which the central-planner should have a greater expected profit than at \( s_1^o(s_2) \).

7. Proof of Theorem 6:

The conclusions reached in each of the three parts of this Theorem can be argued as follows:

1. Consider first the case when \( \alpha_d(s_2) \leq \alpha_0(s_2) \). In this case the supplier chooses \( s_{1\alpha} = 0 \) and since small changes in \( \alpha \) do not affect the component stock level, the manufacturer’s expected profit function is decreasing in \( \alpha > \alpha_d(s_2) \) (it pays more without any change in lead time performance). On the other hand, when \( \alpha_d(s_2) > \alpha_0(s_2) \), Theorem 4 applies and \( z_2 \) is decreasing for any \( \alpha > \alpha_d(s_2) \).

Similarly, the function \( z_1(\alpha, s_{1\alpha}(\alpha, s_2), s_2) \) is increasing in \( \alpha \) (see Theorem 3). Therefore, the manufacturer should offer the minimum revenue fraction necessary to attract the supplier to participate. At this minimum revenue fraction, the manufacturer must also satisfy its participation constraint, otherwise a feasible contract does not exist. These arguments prove item i. of the Theorem.

2. When \( \alpha_d(s_2) > \alpha_0(s_2) \) then \( z_2(\alpha, s_{1\alpha}(\alpha, s_2), s_2) \) is increasing in \( \alpha \) when \( \alpha_0(s_2) \leq \alpha < \alpha_d(s_2) \) and decreasing otherwise. In this case, the manufacturer should offer either \( \alpha_{z_1}(s_2) \) or \( \alpha_d(s_2) \), depending on which results in a greater profit.
3. In the last case of Theorem 6, the unconstrained maximum revenue-fraction is feasible. Hence, the manufacturer chooses $\alpha_d$. 

8. Proof of Theorem 7:
If $\alpha^e(s_2) > k_6(s_2)$ and $\alpha^e(s_2) < \alpha_0(s_2)$, then $s_1^0(s_2) = s_2^0 = 0$ (see condition 32) and the simple revenue-sharing contract achieves coordination. Therefore, we do not consider that case here. There are two cases that remain. If $\alpha^e(s_2) > \alpha_0(s_2) > k_6(s_2)$, then $s_1^0(s_2) > s_2^0 = 0$. Alternatively, if $\alpha^e(s_2) > k_6(s_2) > \alpha_0(s_2)$, then $s_1^0(s_2) > s_2^0 > 0$. From the proof of Theorem 5, we know that $\alpha_d(s_2) > k_6(s_2) > \alpha_0(s_2)$ is not feasible. Also, from Theorem 6, it follows that when $\alpha^e(s_2) > \alpha_0(s_2)$, then $\alpha^e(s_2) \geq \alpha_d(s_2)$. Therefore, $\alpha_e(s_2) > k_6(s_2) > \alpha_0(s_2)$ is not feasible.

Since $s_1^0$ is not the optimal choice of stocking level for the supplier when the manufacturer offers revenue fraction $\alpha^e$, and the supplier’s optimal expected profit is decreasing in $\alpha$, it follows that the minimum revenue-fraction it requires to stock $s_1^0$ exceeds $\alpha^e$, i.e., $\alpha^e(s_2) < \alpha_1(s_1^0(s_2), s_2, z_1^0(s_2))$. Suppose the manufacturer picks a revenue fraction $\alpha^a(s_2)$ such that

$$\alpha_2(s_1^0(s_2), s_2, z_2^0(s_2)) > \alpha^a(s_2) > \alpha_1(s_1^0(s_2), s_2, z_1^0(s_2)).$$

Then, $\alpha^a$ guarantees that both players make at least their equilibrium expected profit $z_1^0$, provided the supplier picks $s_1^0$. This can be achieved by offering $\alpha^a$ in the range $0 \leq s_1 \leq s_1^0$. Since for the supplier $s_{1a}(\alpha^a(s_2), s_2) > s_1^0(s_2)$ (this follows from the fact that $s_{1a}$ is increasing in $\alpha$, $\alpha^a > \alpha^e$, and $s_1^0 > s_1^0$), its expected profit is increasing in $s_1$, for $s_1 \leq s_1^0$, and it will pick the maximum allowed in the range. On the other hand, in the range $s_1 > s_1^0(s_2)$, the manufacturer offers $\alpha^e$ and the supplier picks $s_1^e$.

If $\alpha^e(s_2) < k_6(s_2)$ and $k_6(s_2) \leq \alpha_0(s_2)$, then $s_1^0(s_2) = s_2^0(s_2) = 0$ (see condition 32) and the simple contract coordinates the supply chain. Putting that case aside, and noting that the manufacturer does not pick $k_6(s_2)$, we can make the following claim:

$$k_6(s_2) \geq \alpha_2(s_1^0(s_2), s_2, z_2^0(s_2)) > \alpha^a(s_2) \geq \alpha_1(s_1^0(s_2), s_2, z_1^0(s_2)) > \alpha^e(s_2).$$

That $\alpha_1(s_1^0(s_2), s_2, z_1^0(s_2)) > \alpha^e(s_2)$ follows from the fact that $s_1^0 \neq s_1^e$. The remaining parts of the above inequality can be proved by contradiction as shown below.

- If $k_6(s_2) \leq \alpha_1(s_1^0(s_2), s_2, z_1^0(s_2))$ then it implies that $z_1(k_6(s_2), s_1^0(s_2), s_2) \leq z_1^0(s_2)$ which cannot be true since $k_6(s_2) > \alpha^e(s_2)$ and $z_1(\alpha, s_{1a}(\alpha, s_2))$ is increasing in $\alpha$.

- If $k_6(s_2) \in [\alpha_1(s_1^0(s_2), s_2, z_1^0(s_2)), \alpha_2(s_1^0(s_2), s_2, z_2^0(s_2))]$ then the manufacturer must choose $k_6(s_2)$ over $\alpha^e(s_2)$ since $k_6(s_2)$ yields greater profit for the manufacturer. Clearly, this is not true.

If the manufacturer offers $\alpha^p(s_2) = \alpha^e(s_2)$ for $s_1 < s_1^0(s_2)$, the supplier would still pick its stock level at $s_1^0(s_2)$. That is the unconstrained maximum, which is also feasible since $s_1^0(s_2) < s_1^0$. However, when the manufacturer offers $\alpha^a(s_2)$ for $s_1 \geq s_1^0(s_2)$, the supplier will pick $s_1^0$. This results from the fact that $s_{1a}(\alpha^a(s_2), s_2) < s_1^0(s_2)$ and the supplier’s profit is decreasing in the range $[s_1^0, \infty)$. #