**1 Probability problems**

**1.1 Dice**
Suppose we roll a pair of dice:

1. What is the probability that the second die lands on a higher value than the first die?
2. What is the probability that the sum of the upturned faces is $i$, for $i \in \{2, \ldots, 12\}$?
3. Suppose that we know the sum of the upturned faces is either 5 or 7. What is the probability that the sum is 5?

**1.2 Machine failure**
A system is composed of 5 components, each of which is either working or broken. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector $(x_1, x_2, x_3, x_4, x_5)$, where $x_i = 1$ if component $i$ works and 0 if component $i$ is broken.

1. How many outcomes are in the sample space of this experiment?
2. Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working, or if components 1, 3, and 5 are all working. Let $W$ denote the event that the system works. Specify all the outcomes in $W$.
3. Let $A$ denote the event that components 4 and 5 are broken. How many outcomes are contained in event $A$?
4. How many outcomes are contained in event $A \cap W$?

**2 Conditional probability problems**

**2.1 Witness reliability**
You are a member of a jury judging a hit-and-run driving case. A taxi hit a pedestrian one night and fled the scene. The entire case against the taxi company rests on the evidence of one witness who saw the accident from his window some distance away. He says that he saw the pedestrian struck by a blue taxi. The lawyer for the injured pedestrian establishes the following facts:

1. There are only two taxi colors in town, blue taxis and black taxis. On the night in question, 85% of all taxis on the road were black and 15% were blue.
2. The witness has undergone an extensive vision test and has demonstrated that he can successfully identify the color of a taxi 80% of the time.

Using Bayes’ rule, compute the probability that the car was, in fact, black.
2.2 Socks
Sock drawer $A$ contains 2 white socks and 1 black sock, whereas sock drawer $B$ contains 1 white sock and 5 black socks. A sock is drawn at random from sock drawer $A$ and placed in sock drawer $B$. A sock is then drawn from sock drawer $B$. It happens to be white. What is the probability that the sock transferred was white?

3 Random variables

3.1 Guessing game
One of the numbers 1 through 16 is uniformly chosen at random. You have to guess the number chosen by asking “yes-no” questions. Compute the expected number of questions you will need to ask in each of the three cases:

1. Your $i$th question is “is it $i$?” for $i = 1, \ldots, 16$.
2. Your $i$th question is “is it $Z_i$?”, where $Z_i$ is chosen uniformly at random between 1 and 16. (note: this would be a very stupid way to guess, because we could guess the same wrong guess twice, for example)
3. At each turn, you try to eliminate one-half of the remaining numbers (so that your first question is “is it less than or equal to 8?” and your second question is either “is it less than or equal to 4?” or “is it less than or equal to 12?” depending on the response, and so forth)

3.2 Fitting
The density function of $X$ is given by
\[ f(x) = \begin{cases} 
  a + bx^2 & 0 \leq x \leq 1 \\
  0 & \text{otherwise}
\end{cases} \]
If $E(X) = 3/5$, find $a$ and $b$.

3.3 Machine breakdown
There are two possible causes for a breakdown of a machine. To check the first possibility would cost $C_1$ dollars, and, if that were the cause of the breakdown, the trouble could be repaired at a cost of $R_1$ dollars. Similarly, there are costs $C_2$ and $R_2$ associated with the second possibility. Let $p$ and $1 - p$ denote, respectively, the probabilities that the breakdown is caused by the first and second possibilities. Under what conditions on $p$, $C_i$, $R_i$ (for $i = 1, 2$) should we check the first possible cause of breakdown and then the second, as opposed to reversing the checking order, so as to minimize the expected cost involved in returning the machine to working order?

Note If the first check is negative, we must still check the other possibility.

3.4 A queueing problem
Consider a post office that is staffed by two clerks. Suppose that when Mr. Smith enters the system, he discovers that Ms. Jones is being served by one of the clerks and Mr. Brown by the other. Suppose also that Mr. Smith is told that his service will begin as soon as either Jones or Brown leaves. If the amount of time that a clerk spends with a customer is exponentially distributed with parameter $\lambda$, what is the probability that, of the three customers, Mr. Smith is the last to leave the post office?

3.5 Bus arrivals
Buses arrive at a specified stop at 15-minute intervals starting at 7:00 a.m. That is, they arrive at 7:00, 7:15, 7:30, 7:45, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that he waits

1. less than 5 minutes for a bus
2. more than 10 minutes for a bus.
3.6 Bus breakdown

A bus travels between two cities A and B, which are 100 miles apart. If the bus has a breakdown, the distance from the breakdown to city A has a uniform distribution over (0, 100). There is a bus service station in city A, in B, and in the center of the route between A and B. It is suggested that it would be more efficient to have the three stations located 25, 50, and 75 miles, respectively, from A. Do you agree? Why?

3.7 Gamma distribution (this one’s hard)

If $X$ is an exponential random variable with mean $1/\lambda$, show that

$$E(X^k) = k! / \lambda^k$$

for $k = 1, 2, \ldots$.

**Hint** Write out the expectation

$$E(X^k) = \int_0^\infty x^k f(x) \, dx = \int_0^\infty x^k \lambda e^{-\lambda x} \, dx = \lambda \int_0^\infty x^{k-1} e^{-\lambda x} \, dx$$

Then, apply a change of variables $y = \lambda x$ and use the identity

$$\Gamma(t) = (t-1)! = \int_0^\infty e^{-y} y^{t-1} \, dy$$

4 The normal distribution

4.1 Normal approximation to binomial

Let $X$ be the number of times that a fair coin, flipped 40 times, lands heads. Find the probability that $X = 20$. Use the normal approximation and then compare it to the exact solution.

4.2 Signal transmission

Suppose that a binary message – either 0 or 1 – must be transmitted by wire from location A to location B. However, the data sent over the wire are subject to a channel noise disturbance, so to reduce the possibility of error, the value 2 is sent over the wire when the message is 1 and the value $-2$ is sent when the message is 0. If $m \in \{-2, 2\}$ is the value sent at location A, then $R$, the value received at location B, is given by $R = m + X$, where $X$ is the channel noise disturbance. When the message is received at location B, the receiver decodes it according to the following rule:

- If $R \geq 0.5$, then 1 is concluded.
- If $R < 0.5$, then 0 is concluded.

Assume that $X \sim N(0, 1)$. Compute the probability that an error occurs in decoding (for the case that $m = -2$ and the case that $m = 2$).

5 Reliability theory

5.1 System reliability

Find the reliability of the system shown in figure 1.
Figure 1: A system of components