IE 4521 Midterm Exam 1
Write clearly, and circle your final answer.
On problems 1-5 you must show your work to receive credit.

Name

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Problem 1
Suppose that you are selling soda at a football game and the demand in liters is a random variable with pdf
\[ f(x) = \begin{cases} \text{cx}(100 - x), & 0 \leq x \leq 100 \\ 0, & \text{otherwise} \end{cases} \]

(a) Find the correct value of the constant \( c \).

Solution:
Since \( f \) is a density it must be the case that
\[
1 = \int_{-\infty}^{\infty} f(x) \, dx = c \int_{0}^{100} x(100 - x) \, dx = c \left( 50x^2 - \frac{x^3}{3} \right) \bigg|_{0}^{100}
\]
\[
= c \left( 50 \cdot 100^2 - \frac{100^3}{3} \right) = \frac{500000c}{3},
\]
this gives
\[
c = \frac{3}{500000}.
\]

(b) Suppose you make $.04 per liter of cola sold, and lose $.08 for every unsold liter that you order. If you order 10 liters what is your expected profit?

Solution:
If you order \( L \) liters of cola and the demand is \( x \) then your profit is
\[
P_L(x) = \begin{cases} .04x - .08(L - x), & x \leq L \\ .04L, & x > L. \end{cases}
\]
If \( X \) is the demand in liters and we order \( L \) liters of cola then we need to find \( E[P_L(X)] \). We know that this is
\[
E[P_L(X)] = \int_{0}^{100} P_L(x) f(x) \, dx
\]
\[
= c \int_{0}^{L} (.04x - .08(L - x))(100x - x^2) \, dx + .04cL \int_{L}^{100} (100x - x^2) \, dx
\]
\[
= c \left( .01L^4 - 2L^3 + \frac{2}{3}100^2L \right).
\]
Plugging in \( L = 10 \) into the previous equation gives \( E[P_{10}(X)] = $.389 \)
Problem 2
A new cancer test has been developed. If a person with cancer takes the test they will be positive with .95 probability, and negative with probability .05. If a cancer free person takes the test they will test negative with probability .95 and positive with probability .05. Suppose that the frequency of this cancer in the general population is 1 case per 100000 people. If a person is selected at random and has a positive reaction, what is the probability they have this cancer?

Solution:
Let $A = \{\text{positive test}\}$, $B = \{\text{has cancer}\}$, then from the problem statement we know that $P(B) = 10^{-5}$, $P(A|B) = 0.95$, and $P(A|B') = 0.05$.

The problem asks us to find $P(B | A)$, and since it doesn’t tell us this probability we should use Baye’s rule to calculate

$$
P(B | A) = \frac{P(A | B)P(B)}{P(A|B)P(B) + P(A|B')P(B')}
= \frac{0.95 \times 10^{-5}}{0.95 \times 10^{-5} + 0.05 \times (1 - 10^{-5})}
= 1.9 \times 10^{-4}.
$$
Problem 3

(a) Suppose $X$ and $Y$ are random variables such that $E[X] = 3$, $E[Y] = -5$, $\text{Var}[X] = 4$, $\text{Var}[Y] = 1$. Set $Z = X/2 + 2Y - 3$. Find $E[Z]$ and $\text{Var}[Z]$ if

(i) $X$ and $Y$ are independent

(ii) corr$(X,Y) = 0.3$

Solution
Since the value of $E[Z]$ doesn’t depend on corr$(X,Y)$ we calculate first

$$E[Z] = \frac{1}{2} E[X] + 2E[Y] - 3 = -11.2.$$ 

The variance of $Z$ is given by

$$\text{Var}(Z) = \frac{1}{4} \text{Var}(X) + 4\text{Var}(Y) + (2) (\frac{1}{2}) (2) \text{corr}(X,Y) \sqrt{\text{Var}(X) \text{Var}(Y)},$$

so first if $X$ and $Y$ are independent then corr$(X,Y) = 0$, so for case i) we have $\text{Var}(Z) = 5$, and in case corr$(X,Y) = 0.3$ we get $\text{Var}(Z) = 6.2$.

(b) $X$ and $Y$ are independent Bernoulli random variables with parameter $p$. Let $Z = X + Y$, and $W = |X - Y|$, and find Cov$(Z,W)$.

Solution
Recall that Cov$(Z,W) = E(ZW) - (EZ)(EW)$, so we need to find the three expected values $E(ZW), E(Z)$, and $E(W)$. First $EZ = EX + EY = 2p$.

Next consider $EW = E|X - Y|$, notice that the possible values for $W$ are $\{0, 1\}$ and $P(W = 1) = P(X \neq Y) = 2p(1-p)$, therefore $E(W) = 2p(1-p)$.

Lastly consider $ZW$, and notice that $ZW$ takes values in $\{0, 1\}$ and $P(ZW = 1) = P(X \neq Y) = 2p(1-p)$, so therefore $E(ZW) = 2p(1-p)$.

Combining this information we get

$$\text{Cov}(Z,W) = E(ZW) - (EZ)(EW)$$

$$= 2p(1-p) - 2p2p(1-p)$$

$$= 2p(1-p)(1 - 2p).$$
Problem 4
A fast food restaurant opens at 8am, and the time before the first customer arrives is an exponential random variable with mean 2 minutes.

(a) What is the probability the first customer arrives after 8:05 am?
Solution:
This is just the probability that an exponential random variable with mean 2 is greater than 5, which is
\[ 1 - F(5) = 1 - (1 - e^{-5/2}) = e^{-5/2}. \]

(b) Suppose no customers have arrived by 8:10. What is the probability no customers have arrived by 8:20?
Solution:
From the memoryless property this is just the property that an exponential with mean 2 is greater than 10, and therefore the answer is
\[ 1 - F(10) = 1 - (1 - e^{-5}) = e^{-5}. \]

(c) Suppose that the time between subsequent customer arrivals are independent and also exponentially distributed with mean 2 minutes. What is the distribution for the arrival time of the 100th customer, and how would you approximate this distribution using the central limit theorem?
Solution:
The arrival time of the 100th customer is the sum of 100 i.i.d exponential random variables with mean 2, this is of course a gamma random variable with parameters \( k = 100 \) and \( \lambda = 1/2 \). Via the CLT, we can approximate this distribution with a \( N(200, 400) \) distribution.
Problem 5
Blank DVD’s produced by a certain company will be defective with probability .01, independently of each other. The company sells the DVD’s in packages of size 10 and offers a money-back guarantee that at most 1 of the 10 DVD’s in the package will be defective. If someone buys 3 packages, what is the probability that he or she will return exactly 1 of them? Note do not use a Normal approximation for this problem.

Solution:
Let $D =$ number of defective DVDs in package. Then it follows that $D \sim \text{Bin}(10, .01)$. The event that a package is refundable is the event that $D > 1$, and we know that

$$P(D > 1) = 1 - P(D \leq 1) = 1 - P(D = 0) - P(D = 1),$$

and for $k = 0, \ldots, 10$

$$P(D = k) = \binom{10}{k} (.01)^k (.99)^{10-k}.$$

Therefore

$$P(D = 0) = \binom{10}{0} (.99)^{10} = .9044$$

$$P(D = 1) = \binom{10}{1} (.01)(.99)^9 = .0914,$$

and $P(D > 1) = .0042$ or $P($package is refundable$) = .0042$. If we buy 3 packages, the number of packages that are refundable, $R$, has a $\text{Bin}(3,.0042)$ distribution. Therefore

$$P(R = 1) = \binom{3}{1} (.0042)(1-.0042)^2 = .0125.$$
Problem 6
Mark each of the following statements as True or False. No Need to Show work.
2 point each

(i) $F(x) = e^{-x}, x \geq 0$ is a valid cumulative distribution function.
F: It’s decreasing!

(ii) If $F(x)$ is the cumulative distribution function for a random variable $X$ and $F(1) = F(2)$, then $P(1 < X \leq 2) = 0$.
T: $P(1 < X \leq 2) = F(2) - F(1) = 0$.

(iii) For any events $A$ and $B$ in a sample space, $P(A|B) = 1 - P(A'|B)$.
T: $1 - P(A'|B) = 1 - \frac{P(A' \cap B)}{P(B)} = \frac{P(B) - P(A' \cap B)}{P(B)} = \frac{P(A \cap B)}{P(B)}$.

(iv) If $X$ and $Y$ are independent random variables then $E[e^X \sin(Y)] = E[e^X]E[\sin(Y)]$.
T: Since $X$ and $Y$ are independent random variables then $E[e^X \sin(Y)] = E[e^X]E[\sin(Y)]$.

(v) For a positive integer $\nu$, the $\chi^2_\nu$ is a special case of the beta distribution.
F: The beta distribution has state space of $[0,1]$, the state space of the $\chi^2_\nu$ is $[0, \infty)$, furthermore the $\chi^2_\nu$ is a special case of the gamma distribution.

(vi) If $X_i, 1 \leq i \leq n$ are i.i.d standard Cauchy random variables then for large $n$

$$\frac{1}{n}(X_1 + \ldots + X_n)$$

is approximated by a $N(0, 1/n)$ distribution.
F: The Cauchy random variables don’t have variance so the CLT does not apply.

(vii) If $X_1, X_2, X_3$ are i.i.d. $N(0, 1)$ random variables then

$$\frac{X_1}{\sqrt{(X_1^2 + X_3^2)/2}} \sim t_2$$

F: The denominator and numerator are not independent.

(viii) Let $X$ be a positive memoryless random variable, then for $y > 0$

$$P(X > x + y | X > x) = P(X > x + y).$$

F: The memoryless property states

$$P(X > x + y | X > x) = P(X > y).$$

(ix) If $X$ and $Y$ are random variables with joint pdf

$$f(x, y) = \begin{cases} 
  x + y, & 0 \leq x \leq 1, 0 \leq y \leq 1 \\
  0, & \text{otherwise}
\end{cases}$$

Then $X$ and $Y$ are independent random variables.
F: The joint pdf is not the product of the marginal pdf’s.
(x) If $X \sim \text{Bin}(n, p)$ and $n \geq 20$, $p = 2/n$ then for $k \geq 0$

$$P(X \leq k) \approx \sum_{j=0}^{k} e^{-2} \frac{2^j}{j!}.$$  

T: This is the Poisson approximation.

(xi) If $X \sim \mathcal{N}(0, 1)$ and $Y \sim \mathcal{N}(0, 1)$ then

$$X^2 + Y^2 \sim \chi^2_2.$$  

F: $X$ and $Y$ are not independent.