Before you begin: This exam has 9 pages (including the normal distribution table) and a total of 8 problems. Make sure that all pages are present. To obtain credit for a problem, you must show all your work. If you use a formula to answer a problem, write the formula down. Do not open this exam until instructed to do so.
1. (10 points) Let $E$, $F$, and $G$ be three events in a state space $S$. Write expressions for the events that of $E$, $F$, and $G$

(a) only $E$ occurs

Solution $E \cap F^c \cap G^c$

(b) both $E$ and $G$ but not $F$ occur

Solution $E \cap F^c \cap G$

(c) at least one of the events occurs

Solution $E \cup F \cup G$

(d) at least two of the events occur

Solution $(E \cap F) \cup (E \cap G) \cup (F \cap G)$

(e) all three occur

Solution $E \cap F \cap G$

(f) none of the events occurs

Solution $E^c \cap F^c \cap G^c$

(g) at most one of the events occurs

Solution $(E \cap F^c \cap G^c) \cup (E^c \cap F \cap G^c) \cup (E^c \cap F^c \cap G) \cup (E \cap F \cap G^c)$

(h) at most two of the events occur

Solution $(E \cap F \cap G)^c$

(i) exactly two of the events occur

Solution $(E \cap F \cap G^c) \cup (E \cap F^c \cap G) \cup (E^c \cap F \cap G)$

(j) Simplify the expression $(E \cup F) \cap (F \cup G)$

(hint: draw a Venn diagram)

Solution $F \cup (E \cap G)$
2. (15 points) You ask your neighbor to water a sickly plant while you are on vacation. Without water it will die with probability 0.8; with water it will die with probability 0.15. You are 90 percent certain that your neighbor will remember to water the plant.

(a) What is the probability that the plant will be alive when you return?

**Solution** Let $A$ be the event that your neighbor waters your plant and $B$ the event that your plant is alive. By the law of total probability,

$$\Pr(B) = \Pr(B|A)\Pr(A) + \Pr(B|A^c)\Pr(A^c) = 0.85 \times 0.9 + 0.2 \times 0.1 = 0.785$$

so there is a 78.5% probability that your plant will remain alive.

(b) If the plant is dead, what is the probability that your neighbor forgot to water it?

**Solution** With $A$ and $B$ as before, we have by Bayes’ Rule

$$\Pr(A^c|B^c) = \frac{\Pr(B^c|A^c)\Pr(A^c)}{\Pr(B^c)} = \frac{0.8 \times 0.1}{1 - 0.785} \approx 0.37$$

so there is a 37% probability that your neighbor forgot to water it.
3. (12 points) Let $X$ represent the difference between the number of heads and the number of tails obtained when a coin is tossed $n$ times.

(a) What are the possible values of $X$?

**Solution** If $n$ is even, $X$ can take the values $n + 2k$, for $k \in \{-\lfloor n/2 \rfloor, \ldots, \lfloor n/2 \rfloor\}$.

**Comment** If students only write positive values (i.e. they take the absolute value of the difference), this is OK too.

(b) If the coin is assumed to be fair, what are the probabilities associated with the values that $X$ can take, for $n = 3$?

**Solution** By symmetry of the distribution, $\Pr(X = 3) = \Pr(X = -3) = 1/8$. The only other possible values are $X = 1$ and $X = -1$, and the distribution is symmetric, so $\Pr(X = 1) = \Pr(X = -1) = 3/8$.

4. (13 points) Suppose we flip a fair coin 5 times. Find the probability that:

(a) the first three flips are the same

**Solution** $(1/2)^3 + (1/2)^3 = 1/4$ (all heads or all tails)

(b) either the first three flips are the same, or the last three flips are the same

**Solution** Let $A$ denote the event that the first three flips are the same and $B$ the event that the last three flips are the same. Then $\Pr(A) = \Pr(B) = 1/4$, and $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$. The event $A \cap B$ happens if all five flips are heads or all five flips are tails, and therefore $\Pr(A \cap B) = (1/2)^5 + (1/2)^5 = 1/16$. Therefore, we find that

$$\Pr(A \cup B) = 1/4 + 1/4 - 1/16 = 7/16$$
5. (15 points) A random variable $X$, which represents the weight (in ounces) of an article, has a density function given by $f(x)$ with

$$f(x) = \begin{cases} 
x - 8 & \text{if } 8 \leq x \leq 9 \\
10 - x & \text{if } 9 \leq x \leq 10 \\
0 & \text{otherwise}
\end{cases}$$

(a) Calculate the mean and variance of $X$.

**Solution** We have

$$E(X) = \int_{-\infty}^{\infty} x f(x) \, dx$$
$$= \int_{8}^{9} x(x - 8) \, dx + \int_{9}^{10} x(10 - x) \, dx$$
$$= 9$$

$$E(X^2) = \int_{8}^{9} x^2(x - 8) \, dx + \int_{9}^{10} x^2(10 - x) \, dx$$
$$= 487/6$$

and therefore

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{487}{6} - 9^2 = 1/6$$

(b) If an article has weight $X$, then the distributor will sell it for $\frac{X}{2}$. He guarantees to refund the purchase money to any customer who finds the weight of his article to be less than 9 oz. An article costs $2.00 to produce. Find the expected profit per article.

**Solution** Let $g(X)$ denote the profit associated with an article of weight $X$. If $X \leq 9$, then the distributor’s profit is $-2$ dollars, since the customer returns the article. If $X > 9$, then the distributor earns a profit of $\frac{X}{2} - 2$ dollars, and therefore

$$g(X) = \begin{cases} 
-2 & \text{if } X \leq 9 \\
\frac{X}{2} - 2 & \text{if } X > 9
\end{cases}$$

We find that the expected profit is

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x) \, dx$$
$$= \int_{8}^{9} (-2)(x - 8) \, dx + \int_{9}^{10} (\frac{x}{2} - 2)(10 - x) \, dx$$
$$= 1/3 \approx $0.33$$

6. (15 points) In a typical sports playoff series, two teams play a sequence of games until one team, the winner, has won four games. Suppose that in each game, team $A$ beats team $B$ with a probability of 0.55, and that the results of different games are independent.

(a) Explain how the binomial distribution can be used to analyze this problem.

**Solution** The outcome of each game can be regarded as a Bernoulli random variable $X_i$, with $X_i = 1$ indicating a win for team $A$ and $X_i = 0$ a win for team $B$. Hence, if $X$ is a binomial random variable with $p = 0.55$ and $n = 7$, then a win for $A$ corresponds to $X > 3$ and a win for $B$ corresponds to $X \leq 3$.

(b) Explain how the negative binomial distribution can be used to analyze this problem.

**Solution** Let $X$ be a negative binomial random variable with $p = 0.55$ and $r = 4$. Then a win for $A$ corresponds to $X \leq 7$ and a win for $B$ corresponds to $X > 7$. Furthermore, if $X \leq 7$, then the value of $X$ tells us exactly how many games it took team $A$ to win the tournament.
(c) What is the probability that team A wins the series in game seven?

**Solution** By part (b) this is just \( Pr\left(X = 7\right) = C\left(7 - 1, 4 - 1\right) (0.45)^3 (0.55)^4 \approx 0.167 \)

(d) What is the probability that team A wins the series in game six?

**Solution** By part (b) this is just \( Pr\left(X = 6\right) = C\left(6 - 1, 4 - 1\right) (0.45)^2 (0.55)^4 \approx 0.185 \)

(e) What is the probability that team A wins the series?

**Solution** This is just \( Pr\left(X = 4\right) + Pr\left(X = 5\right) + Pr\left(X = 6\right) + Pr\left(X = 7\right) \approx 0.608 \)

7. (15 points) Suppose that \( X \sim \mathcal{N}(64, 25) \).

(a) What is \( Pr\left(59.9 \leq X \leq 70.0\right) \)?

**Solution** We have

\[
Pr\left(59.9 \leq X \leq 70.0\right) = Pr\left(\frac{-4.15}{5} \leq \frac{X - 64}{5} \leq \frac{6.0}{5}\right)
\]

\[
= Pr\left(-0.82 \leq Z \leq 1.2\right)
\]

\[
= \Phi(1.2) - \Phi(-0.82) \approx 0.679
\]

(b) What is \( Pr\left(|X - 64| \leq 3\right) \)?

**Solution** We have

\[
Pr\left(|X - 64| \leq 3\right) = Pr\left(61 \leq \frac{X - 64}{5} \leq 67\right)
\]

\[
= Pr\left(-0.6 \leq \frac{X - 64}{5} \leq 0.6\right)
\]

\[
= \Phi(0.6) - \Phi(-0.6) \approx 0.4515
\]

(c) Find \( y \) so that \( Pr\left(X \leq y\right) \approx 0.30 \).

**Solution** We have

\[
Pr\left(\frac{X - 64}{5} \leq \frac{y - 64}{5}\right) = 0.30
\]

\[
Pr\left(Z \leq \frac{y - 64}{5}\right) = 0.30
\]

We find that \( z \approx -0.52 \) and therefore \( y = 5z + 64 \approx 61.4 \).
8. (10 points) True/false. No justification is needed.

(a) If the conditional probability $\Pr (A|B)$ is equal to $\Pr (A)$, then the events $A$ and $B$ are independent. True.

(b) If $\Pr (A|B) = \Pr (A)$, then $\Pr (B|A) = \Pr (B)$. True.

(c) The event $A$ and its complement $A^c$ are independent. False.

(d) SAT-style analogy: Exponential distribution:Gamma distribution :: Geometric distribution:Negative binomial distribution. True.

(e) If $X$ is a memoryless random variable, then $\Pr (X \geq x + y|X \geq x) = \Pr (X \geq y)$. True.

(f) If $Z \sim \mathcal{N}(0,1)$, then $E(Z^2) = 1$. True.

(g) If $X \sim \mathcal{N}(0,1)$ and $Y \sim \mathcal{N}(0,1)$, then $X + Y \sim \mathcal{N}(0,1)$. False.

(h) If $X \sim \mathcal{N}(0,4)$ and $Y \sim \mathcal{N}(0,1)$, then $X - Y \sim \mathcal{N}(0,3)$. False.

(i) If $\Pr (A) = 0.55$ and $\Pr (B) = 0.65$, then $A$ and $B$ cannot be mutually exclusive. True.

(j) If $X_1, \ldots, X_{100}$ are independent random variables with mean 0 and variance 10, then $\frac{X_1 + \cdots + X_{100}}{100}$ roughly follows a normal distribution with mean 0 and variance 0.1. True.