A pricing model for clearing end-of-season retail inventory

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Abstract

The problem of setting prices for clearing retail inventories of fashion goods is a difficult task that is further exacerbated by the fact that markdowns enacted near the end of the selling season have a smaller impact on demand. In this article, we present discrete-time models for setting clearance prices in such an environment. When demand is deterministic, we compute optimal prices and show that decreasing reservation prices lead to declining optimal prices. When demand is stochastic and arbitrarily correlated across planning periods, we obtain bounds on the optimal expected revenue and on optimal prices. We also develop a heuristic procedure for finding near-optimal prices and test its accuracy through numerical experiments. These experiments reveal new insights for practitioners. For example, the penalty for choosing clearance price once and keeping it unchanged for the remainder of the selling season is found to be small when either the mean reservation prices do not change appreciably over time or when they drop sharply after the first period.
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1. Introduction

With the rapid proliferation of retail SKUs (stock-keeping units), mitigating market mediation costs in the retail of fashion goods has become increasingly important for profitability of retailers. When facing
less-than-expected sales, retailers try to recover as much revenue as possible via price markdowns. Mistakes can be very costly. For example, one large web retailer (Amazon.com) reported a $39 million charge in 1999 for unsold Christmas inventory (Vogelstein, 2000). OfficeMax reported a “special markdown charge” of approximately $50 million to liquidate inventory in order to accelerate profitability (OfficeMax, 1999). The Chief Executive Office of Ann Taylor Stores reported that 1996 markdown units were more than 40% of the inventory (Ann Taylor Stores Press Release, 1977). Smith and Achabal (1998) report that the difference between the regular-price and the actual-sales dollars is “often several hundred million dollars for major retailers.”

A retailer must address two types of questions when faced with excess inventory near the end of the regular selling season: timing the markdown (or markdowns) and deciding how much discount to offer at each such decision epoch. The clearance period is bounded by the first markdown and the “outdate” when all remaining inventory is salvaged and new items arrive to replace the old ones on the store shelves. These problems are exacerbated by the fact that it becomes increasingly more difficult to affect demand via discounts as the selling season progresses.

For large retail chains that have multiple outlets, clearance prices can be different at each store, based on local inventory and forecasted demand during the clearance period. This practice supports the observation that competitive response to price markdown is not significant (Zabel, 1970). Therefore, models used for setting clearance prices typically do not account for competition.

There is a large body of literature on the problem of setting prices for perishable inventories. In what follows, we shall examine selected studies in terms of problem characteristics and model assumptions first, followed by a review of their main contributions. We will then compare and contrast our models with those in other articles. Since a review of the relevant literature can be found in the first section of Zhao and Zheng (2000), the ensuing discussion is intended to help position our study in relation to the literature. It is not intended to be exhaustive.

In the pricing-inventory literature, one of two approaches is used to model demand: either a demand function is assumed to be given exogenously, or else a consumer-choice model is constructed that gives rise to the demand function. When demand is modeled directly, it typically belongs to the following class of functions (see an exception described in the next paragraph): \( D_t(p) = \gamma_t(p) \xi_t + \delta_t(p) \), where \( D_t \) is the random demand at time \( t \), \( \gamma_t() \) and \( \delta_t() \) are non-increasing functions of price \( p \) and \( \xi_t \) is a random variable. We mention two articles belonging to this class. Both consider discrete-time models. Zabel (1972) treats the multiplicative and additive demand models separately; the multiplicative model is realized by setting \( \delta_t(p) = 0 \), and the additive model by setting \( \gamma_t(p) = 1 \). In each case, the random component is time-independent and can have either Uniform or Exponential distribution. Federgruen and Heching (1999) work with the more general demand function, permit additional purchases during the planning horizon, and allow an arbitrary dependence of demand on price and time. The random component of demand in their model can have an arbitrary distribution; however mean demand must be non-increasing and concave in \( p \). Federgruen and Heching’s work subsumes a number of other articles pertaining to this class of problems. Other related papers are Chen and Simchi-Levi (2002) and Chan et al. (2001).

Smith and Achabal (1998) develop a deterministic model that generalizes the demand function mentioned above by allowing demand to depend on inventory level, in addition to price and time. Lower inventory indicates smaller assortment, which can retard demand if it falls below a minimum inventory level that they call the “fixture-fill” quantity. The model allows prices to be different at each store, based on local conditions such as inventory levels and demand rates.

In contrast, studies that model consumer behavior assume that demand materializes when a customer browses the store and finds the price of the item to be smaller than her reservation price at that point in time. Suppose the intensity with which customers arrive to browse the store at time \( t \) is \( \lambda(t) \) and the reservation price distribution of a customer arriving at time \( t \) is \( F_t() \). Then, the probability that the customer will buy the item offered at price \( p \) is \( u(p,t) = 1 - F_t(p) \) and the demand rate at \( t \) is \( \lambda(t)u(p,t) \). Within this
framework, Gallego and Van Ryzin (1994) consider a continuous-time model in which the demand rate is time-invariant and given by \( \lambda_t = \lambda (1 - F(p)) \).

In the same paper, Gallego and Van Ryzin further extend their analysis to the case when demand intensities are non-homogeneous. Specifically, they assume that \( \lambda_t = \lambda(p)g(t) \), with \( \lambda(p) \) and \( g(t) \) being known functions of price and time, respectively.

Gallego and Van Ryzin (1997) consider multiple products. Item \( j \) demand is a continuous-time point process with time-dependent Markovian intensity \( \lambda_j(p,t) \), giving rise to a multivariate Poisson demand process. Batch arrival of demand in the form of a compound Poisson process can also be handled through this approach. However, demand arises from an underlying consumer-choice model in which reservation prices are i.i.d. and time-invariant. In contrast, Bitran and Mondschein (1997) allow both \( \lambda(t) \) and \( F_j \) to be dependent on time in an arbitrary fashion. This subsumes cases such as the retail of fashion goods in which customer utility is highest at the start of the selling season and declines thereafter, as well as cases such as the sale of seats on a flight leg where customer utility is generally low earlier in the selling season. Bitran et al. (1998) describe an application of the model presented in Bitran and Mondschein to a large retail chain. Both are discrete-time models. Zhao and Zheng (2000) develop a continuous-time analog of time-sensitive demand and reservation-price model.

It is easy to see that the demand models assumed by Zabel (1972), Federgruen and Heching (1999), and others, can arise from an appropriately constructed consumer-choice model. To illustrate this point, consider the demand function \( D_i(p) = K_i e^{-b_i p} \). Note that this is the demand function we shall work with later in this article. Let \( K_i E(\xi_i) \) be the mean arrival rate of customers into the store, and assume that the reservation prices have an exponential distribution with an average reservation price of \( \beta_i^{-1} \). Then, \( u(p,t) = e^{-b_i p} \) and we obtain the above-mentioned demand function. Another feature that is common to all of the articles mentioned above is that the demand in any time interval/period is independent of the demand in other intervals/periods.

We next turn to the main contributions of the above-mentioned articles. The papers by Gallego and Van Ryzin (1994); Gallego and Van Ryzin (1997); Bitran and Mondschein (1997) and Zhao and Zheng (2000) establish the existence of two structural properties: optimal price decreases in the number of items left, and given a fixed number of items, the optimal price decreases over time. These articles also present algorithms and heuristics for determining the optimal price trajectory. For example, when the demand intensity is an exponential function of price, Gallego and Van Ryzin (1994) show that the optimal price obtained from the deterministic model is asymptotically optimal for the stochastic model. The optimal price in their deterministic model is a fixed price for the entire selling season. Federgruen and Heching (1999) show that the optimal policy is a list-price, order-up-to policy. For each list price, there exists an order-up-to level such that if the current stock level is below the order-up-to level, new items are ordered; otherwise, the list price is reduced.

In this paper, we consider a discrete-time demand function of the type \( D_i(p) = K_i e^{-b_i p} \). That is, both customer arrival intensity and reservation price are time-dependent and the random component of the demand function can have an arbitrary distribution. In this sense, our demand function is similar to the functions described in Bitran and Mondschein (1997) and Zhao and Zheng (2000). However, this demand function is not in the class of functions allowed in the Federgruen and Heching (1999) paper since mean demand in our model is non-increasing and convex in price. (Recall that Federgruen and Heching require the mean demand to be concave in price.) The former construct is the more natural choice for modeling how price affects demand. It is also frequently used in the economics literature. Another important feature of our model is that we allow demand in different periods to be arbitrarily correlated through \( \xi_i \). The resulting problem is difficult to solve since we can no longer use the MDP (Markov Decision Processes) methodology that underlies all of the articles mentioned above. Consequently, we obtain upper and lower bounds on the optimal expected revenue and optimal prices, and develop a heuristic procedure to solve for optimal prices. The accuracy of this heuristic is tested in numerical experiments. By assuming a special
(exponential) distribution of reservation prices, we are able to obtain sharper insights through numerical experiments. An example is presented below for illustration.

Retail managers prefer not to change prices at each decision epoch. In fact, a single pricing policy in which the clearance price is set once and left unchanged for the remainder of the clearance period is often used. Such actions are prompted by the fact that fewer price changes are easier to implement and avoid the cost of relabeling merchandise. Naturally, it is important to know whether the single pricing strategy is substantially suboptimal, especially when mean reservation prices decline over time. Specifically, for what values of the problem parameters is it appropriate for retail managers to use a single markdown strategy and when is it inappropriate? We address these and other managerially important issues through numerical experiments.

In the models reported here, we are not concerned with how much inventory should be ordered at the start of the selling season. Instead, our focus is on determining optimal prices. However, at least when demand is deterministic our approach leads to an algebraic expression for finding optimal prices for each given starting inventory level. Therefore, in a deterministic setting, our model can be used to develop an efficient search technique for finding the optimal stocking quantity, with contingent pricing decisions being made optimally. Also, the models can be used to find the start of the markdown portion of the selling season by modeling the entire selling season and marking the period in which the first markdown occurs. Thus, we address a class of problems involving pricing decisions for finite, non-replenishable inventory when demand arrival intensity and reservation prices are time-dependent, and demand is correlated across time.

Before closing this section, we mention some areas that are related to the problem we model, but where the critical features of the problem are somewhat different. For example, whereas the clearance pricing problem is modeled without accounting for competition, there are several papers that consider pricing decisions under competition. Dockner and Jorgensen (1988) provide a treatment of the optimal pricing strategies for oligopolistic markets from a marketing perspective. Other recent papers include Bernstein and Federgruen (1999) who develop a stochastic general equilibrium inventory model for supply chains in an oligopoly environment; and Perakis and Sood (2003) who analyze the competitive dynamic pricing assuming general demand models and propose a general algorithm for computing equilibrium prices.

Another related class of problems concerns pricing and inventory decisions in a single period (newsvendor) setting. On the supply side, Jucker and Rosenblatt (1985) studied the newsvendor problem with a single demand with quantity discounts for purchasing costs. On the demand side, Khouja (1995) has considered the newsvendor problem when it contains progressive discounts to sell off excess inventory and further extended his earlier model (Khouja, 1996) to include multiple discounts from the supplier side. Sen and Zhang (1999) study the single-item newsvendor problem where the item can be sold to different demand classes (realized sequentially over time) at different prices; they show that aggregating the multiple demands with a single average price or applying the single demand model separately to multiple demand classes may lead to significant suboptimality. Literature in this area has been summarized by Petruzzi and Dada (1999).

There is a close parallel between clearance pricing of retail inventories and pricing of perishable inventory of seats on an airplane or rooms in a hotel. The latter problems belong to the class of perishable asset revenue management (PARM) problems that have been the focus of considerable research activity in recent years; see McGill and van Ryzin (1999) for a review.

This paper is organized as follows. Section 2 develops the deterministic-demand and stochastic-demand clearance pricing models assuming a single pricing opportunity. Section 3 explores models with multiple opportunities for setting prices. Section 4 presents numerical examples and managerial insights. In Section 5, we focus on identifying characteristics of some markets that make them provably more desirable than others, provided that the pricing decisions are made optimally.

Contributions of this paper are summarized in Section 6.
2. Single clearance-pricing opportunity

In this section, we consider the case in which the pricing decision is made only once at the start of the clearance period, and once selected, this price is not changed until the management-specified outdate occurs. The store manager’s objective is to maximize expected revenue by choosing price \( p \), given initial inventory level \( I \), and a salvage value \( s \) (net of holding and disposal charges) for any leftover items at the outdate. Recall from the previous section that demand has the following functional form: \( D(p) = y(p) \xi \), where \( y(p) = Ke^{-\beta p} \) is called the price-demand model, and \( \xi \) is the multiplicative random variable that represents the error term. In this model, \( K \) is a measure of the size of the market—the amount that will be sold if price is set equal to zero. Alternatively, \( K\xi(\xi) \) can be interpreted as the mean number of customers that browse the store. Parameter \( \beta \) determines the sensitivity of demand to price. It is also the inverse of mean reservation price. Larger \( \beta \) implies smaller mean reservation price, which means that a greater discount will be needed to tempt customers to buy the product.

Quantities of the item under consideration are continuous. Parameters \( \mu \) and \( \sigma^2 \) are the mean and variance of \( \xi \), which is assumed to be continuous. We use \( f(\cdot) \) and \( F(\cdot) \) to denote the probability density and cumulative distribution functions of \( \xi \). For the deterministic-demand model, we use \( d(p) = \mu y(p) \) to denote demand. It is implicitly assumed that irrespective of the amount of leftover stock, all excess inventory can be salvaged at the fixed salvage \( s > 0 \) per unit. This is quite realistic because retailers have the option of either selling their leftover inventory to discounters, or donating to charity, or discarding. The latter two options generate a positive cash flow through a tax write-off. We use \( \pi(\cdot) \) to denote the revenue function, with superscripts “\( d^* \)” and “\( s^* \)” signifying deterministic-demand and stochastic-demand scenarios respectively.

The results presented in this section are similar to the analysis by Petruzzi and Dada (1999), except that their price-demand model is a power function. That is, using our notation, Petruzzi and Dada use the function \( y(p) = ap^{-b} \), where \( a, b > 0 \) are real numbers. We include the analysis of the one-period problem since it provides a foundation for the more complicated and realistic model with multiple-markdowns. The analysis is presented in two parts. The first part deals with deterministic demand and the second with stochastic demand.

2.1. Single pricing opportunity with deterministic demand

In this deterministic or riskless environment, if the retailer picks a price \( p \), it knows with certainty that the demand will be \( d(p) = \mu y(p) \). Clearly, the retailer should never pick a price such that \( d(p) > I \). This is because in every such situation it can extract extra revenue by charging a higher price for which \( d(p) = I \). Therefore, we wish to solve the following optimization problem to find the optimal price \( p \) that maximizes total revenue.

\[
\text{Maximize} \quad \pi^d(p) = pd(p) + s(I - d(p)),
\]

subject to:

\[
p : d(p) \leq I.
\]

The objective function does not contain a cost term signifying the purchase cost to the retailer. This is consistent with industry practice and underscores the fact that the original purchase cost is sunk. Constraint 2 simplifies to yield \( p \geq -\ln(I/\mu K)/\beta \). Since \( p \) must also be at least \( s \), the constraint is meaningful only if \( -\ln(I/\mu K)/\beta \geq s \), i.e., when there exists a price level \( p > s \) at which the retailer could sell its entire surplus inventory \( I \).

Setting the first derivative of \( \pi^d(p) \) to zero, we find that it has two roots: either \( p = 1/\beta + s \) or \( p = \infty \). It is easy to verify that the second derivative of \( \pi^d \) is negative only when \( p < 2/\beta + s \) which means that only
\( p = 1/\beta + s \) is a local maximum. Therefore, the optimum price is the unconstrained solution, \( p = 1/\beta + s \) if \( \mu y(1/\beta + s) \leq I \), and \( p = -\ln(I/\mu K)/\beta \) otherwise. Put differently, set \( p = 1/\beta + s \) unless upon doing so the corresponding demand exceeds \( I \), in which case set price at the higher value of \(-\ln(I/\mu K)/\beta \). The latter is also called the stock-clearing price. These arguments are summarized in the following proposition.

**Proposition 1.** The optimal riskless price \( p^0 \) is given as

\[
p^0 = \max \{ 1/\beta + s, -\ln(I/\mu K)/\beta \}.
\]

It is easy to verify that \( \pi^d(p) \) is increasing in \( I \) for each \( p \), and that \( p^0 \) is decreasing in \( \beta \) and \( I \).

Problem (1 and 2) can also be cast in terms of the equivalent quantity variable, say \( x \). If we define the inverse demand function, \( p(x) = (1/\beta)\ln(K/y) \), to be the price at which all \( x \) units can be sold, then the store manager’s problem in equivalent quantity variable terms is:

\[
\max p_d(x) = yp(x)
\]

subject to:

\[
x \leq I,
\]

\[
x \geq 0.
\]

We continue to work throughout this article with the formulation involving prices as decision variables since that has greater intuitive appeal for the store manager. Note that proofs of some propositions can be simplified by using the formulation in (3)–(5).

### 2.2 Single pricing opportunity with stochastic demand

Let \( z(p) = I/y(p) \) be the realization of random component of demand for which the entire inventory can be sold at price \( p \). Note that \( z(p) \) is completely determined by \( p \). However, we suppress its argument in the ensuing discussion for notational compactness and note that any function of \( z \) is an implicit function of \( p \).

The revenue function can now be written as follows:

\[
\pi^d(p, \xi) = \begin{cases} pD(p) + s(I - D(p)) & \text{if } \xi \leq z, \\ pI & \text{otherwise.} \end{cases}
\]

Setting \( I = y(p)z \) in (6), taking expectations over all values of \( \xi \), and simplifying, we obtain the following expected revenue function:

\[
E[\pi^d(p)] = sI + y(p)(p - s)(\mu - \Theta(z)),
\]

where \( \Theta(z) = E(\xi - z)^+ = \int_z^\infty [u - z]dF(u) \). The expected revenue function can be written as the sum of two functions: a riskless revenue function and a loss function which is caused by demand uncertainty. That approach leads to the following representation: \( E[\pi^d(p)] = \pi^d(p) - L(p) \), where \( \pi^d(p) \) is as given in (1) and \( L(p) = y(p)(p - s)\Theta(z) \) is the expected loss due to demand uncertainty. Applying Leibnitz’s rule to differentiate (7), we obtain our first major result shown below in Theorem 1 (see Appendix A for proof).

**Theorem 1.** The optimal price under stochastic demand \( p^* \) is obtained by solving the following equation:

\[
p^* = \frac{\nu(z^*)}{\beta} + s,
\]

where \( \nu(z) = [\mu - \Theta(z)]/[\mu - \Theta(z) - z(1 - F(z))] \geq 1 \), and \( z^* = I/y(p^*) \).
An immediate consequence of Theorem 1 is that \( p^* \geq p^0 \). This inequality can be explained by noting that the variance of demand is decreasing in \( p \) whereas its coefficient of variation is independent of \( p \). In fact, the optimal price in a stochastic demand environment exceeds the optimal riskless price in all multiplicative demand models (see Petruzzi and Dada, 1999, for details). Since \( z \) is a function of \( p \), the optimal price is obtained via a numerical solution of Eq. (8), and the optimal expected revenue is obtained by substituting \( p^* \) into (7).

Analogous to the deterministic demand problem, we can show that \( E[\pi'(p)] \) is increasing in \( I \) and that \( p^* \) is decreasing in \( I \) and \( \beta \). These results match claims made in earlier papers and are satisfying on an intuitive level since we expect larger amount of surplus stock \( I \), and/or smaller mean reservation price (larger \( \beta \), to be reasons for setting a lower price. That \( E[\pi'(p)] \) is increasing in \( I \) can be seen directly by examining (7). In order to verify that \( p^* \) is decreasing in \( I \) and \( \beta \), recall first that \( p \) and \( z \) are related through the relationship \( z = \frac{I}{y(p)} \), and that \( \frac{dp}{dz} = \frac{p}{y(p)} = pz > 0 \). Next, we use \( F^{(1)}(z) \) to denote \( \int_0^z F(u) du \), and then differentiate \( v(z) \) with respect to \( z \), which results in the following observations:

\[
\frac{dv}{dz} = z[F(z)(1 - F(z)) - zf(z)] + F^{(1)}(z)[zf(z) - (1 - F(z))]
\]

\[
\leq z[F(z)(1 - F(z)) - zf(z)] + zF(z)[zf(z) - (1 - F(z))]
\]

\[
= z^2 f(z)[F(z) - 1] \leq 0.
\]

The first inequality in (9) follows from the fact that \( F(u) \) is an increasing function of \( u \), for all \( u \geq 0 \). Therefore \( F^{(1)}(z) \leq z F(z) \) for all \( z \geq 0 \). Finally, \( \frac{dv}{dz} = \frac{z f(z)}{F(z)[F(z) - 1]} \leq 0 \), and a similar series of steps imply that \( \frac{dp}{dz} \leq 0 \).

At this stage, it is useful to summarize insights resulting from the analysis of the single clearance-pricing problem. These help guide our treatment in the next section. We observe that the optimal price is higher in the stochastic demand model. Also, for each price \( p \), \( \pi'(p) \geq E[\pi'(p)] \). In particular, this holds for \( p = p^* \), which leads to the following conclusion: \( E[\pi'(p^0)] \leq E[\pi'(p^*)] \leq \pi'(p^*) \leq \pi'(p^0) \). Another important observation is that \( \pi'(p) \) and \( E[\pi'(p)] \) are increasing in \( I \), whereas \( p^0 \) and \( p^* \) are decreasing in both \( I \) and \( \beta \).

### 3. N-Pricing opportunities

We now consider the situation in which there are \( N \) opportunities to set prices for clearance items, once at the start of each of \( N \) periods. We use subscript \( n \) to represent the \( n \)th period, where \( n = 1, 2, \ldots, N \). Thus, the surplus inventory at the start of the \( n \)th period in the selling season is denoted by \( I_n \). Bold font is used for vectors. For example \( p = (p_1, \ldots, p_N) \) represents the vector of prices, and \( d = (d_1, \ldots, d_N) \) the vector of demands. (Note that the appropriate notation for period-\( n \) demand is \( d_n(p_n) \) when demand is constant, and \( D_n(p_n) \) when demand is random. However, we suppress the argument \( p_n \) in order to make the notation compact.) In practice \( N \) is small, with usually no more than two or three opportunities to adjust prices. Furthermore, since the entire selling period is relatively small (typically 4–13 weeks), we do not discount revenues from different periods and do not include inventory holding costs in the objective function.

#### 3.1. N-Pricing opportunities with deterministic demand

Analogous to (1), the \( n \)th period revenue function, which is denoted by \( \psi_n \), can be written as follows:

\[
\psi_n(p_n) = \begin{cases} 
  p_n \min\{I_n, d_n\} & \text{if } n < N, \\
  p_N \min\{I_N, d_N\} + s_N(I_N - d_N)^+ & \text{otherwise},
\end{cases}
\]
where \( I_n = (I_{n-1} - d_{n-1})^+ \), for each \( n \geq 2 \), \( d_n = \mu_n y_n(p_n) \), \( y_n(p_n) = K_n e^{-b_{nm}} \), and \( s_N \) is the salvage value. Summing both sides of (10) for all \( n \) from 1 through \( N \), and simplifying, we obtain the following expression for \( \pi^d(p) \):

\[
\pi^d(p) = p_1 I_1 - \sum_{n=1}^{N} (p_n - p_{n+1}) (I_n - d_n)^+ ,
\]

where \( p_{N+1} \equiv s_N \). Note that \((I_n - d_n)^+\) is zero if \( I_1 < \sum_{i=1}^{n} d_i \), and \((I_1 - \sum_{i=1}^{n} d_i)\) otherwise. Therefore (11) can be written alternatively as follows:

\[
\pi^d(p) = p_1 I_1 - \sum_{n=1}^{N} (p_n - p_{n+1}) \left( I_1 - \sum_{i=1}^{n} d_i \right)^+ .
\]

Let \( p^0 = (p_1^0, \ldots, p_N^0) \) be the optimal price vector for the deterministic problem, i.e., \( p^0 = \arg\{\max \pi^d \} \). Solving for \( p^0 \) is made difficult by the presence of the \((I_1 - \sum_{i=1}^{n} d_i)^+\) terms. However, the problem becomes trivial in one special case. Suppose there is ample inventory to meet demand in each period, irrespective of the prices chosen. In that case, the problem of setting prices decouples into \( N \) separate one-period problems. From Proposition 1, the corresponding optimal period-\( n \) price is \( p_n = \frac{1}{b_{ni}} + s_N \). Clearly, this solution is optimal overall if \( \sum_{i=1}^{N} p_i y_i \left( \frac{1}{b_{ni}} + s_N \right) \leq I_1 \). That is, when the preceding inequality holds, we have the optimal price vector without further effort. Therefore, in the remainder of this subsection, we assume that \( \sum_{i=1}^{N} p_i y_i \left( \frac{1}{b_{ni}} + s_N \right) > I_1 \). Our approach is to prove certain properties of the optimal price vector that lead to quick and efficient method for solving the problem. These same properties also prove to be useful for developing a heuristic solution procedure for the stochastic-demand analog of the model considered here.

**Observation 1.** For each period \( n \), \( 1 \leq n \leq N \), the price \( p_n \) should be chosen so that either \( d_n = I_n \), or \( d_n < I_n \). In other words, \( p_n \) should never be chosen so as to make \( d_n > I_n \).

It is easy to verify that the above observation must hold. Consider the situation in which \( I_n > 0 \) and the price in period \( n \) is chosen to make \( d_n > I_n \). In every such case, the retailer can increase revenue by keeping prices in period 1 through \( n-1 \) unchanged and simply increasing price in period \( n \) until \( d_n = I_n \). Notice that since \( I_j = 0 \) for all \( j > n \), periods \( n+1 \) through \( N \) do not affect revenue. An immediate consequence of this observation is that we can impose the following structure on the choice of optimal price vector:

**Proposition 2.** The price vector \( p^0 \) must be such that one of the following is true:

(i) For some \( n \), where \( 1 \leq n \leq N \):

\[ \sum_{j=1}^{n} d_j = I_1, \quad I_j = I_1 - \sum_{i=1}^{j-1} d_i > 0 \quad \text{for } j = 1, 2, \ldots, n, \quad \text{and} \quad I_j = 0, \quad \text{for all } j > n. \]

(ii) \( I_1 > \sum_{j=1}^{N} d_j. \)

Thus, in order to find \( p^0 \), we need to solve at most \( N \) deterministic problems such that in the \( n \)th problem, period \( n \) is the first period in which \( d_n = I_n \). For the \( n \)th problem, we need to find \( n-1 \) optimal prices. The price in period \( n \) is determined uniquely by the fact that \( \sum_{j=1}^{n} d_j = I_1 \) and prices in periods \( n+1 \) through \( N \) do not matter since \( I_j = 0 \) for all \( j > n \). For simplicity, we set these to zero. The \( n \)th deterministic problem, with objective function \( \pi^d_n \), can be written as follows:

\[
\text{Maximize} \quad \pi^d_n(p) = p_1 I_1 - \sum_{j=1}^{n-1} (p_j - p_{j+1}) \left( I_1 - \sum_{i=1}^{j} d_i \right),
\]
subject to:

\[ \sum_{i=1}^{j} d_i - I_1 \leq 0, \quad \text{for all } j = 1, 2, \ldots, n-1, \text{ and,} \]

\[ \sum_{j=1}^{n} d_j = I_1. \]

(14)  (15)

We further simplify the solution procedure by utilizing the following result with its proof shown in Appendix B.

**Proposition 3.** Let \( p_0^n \) denote the solution to the \( n \)th problem described in (13)–(15). Then, \( \pi^d(p_0^n) \geq \pi^d(p_0^0) \) for all \( n \).

That is, of the \( N \) possible cases, \( N-1 \) are dominated, leaving exactly one candidate. The problem formulated in (13)–(15) can be solved as explained below.

**Theorem 2.** The optimal price vector for \( N \)th deterministic problem can be obtained as follows:

\[ p_N : I_1 = \sum_{j=1}^{N} \mu_j K_j e^{-\beta_j p_j}, \]

\[ p_n = p_N + \frac{1}{\hat{p}_n} - \frac{1}{\hat{p}_N}, \quad \text{for all } n = 1, 2, \ldots, N-1. \]

(16)  (17)

Proof of Theorem 2 is presented in Appendix C. An efficient procedure for solving the clearance pricing problem with \( N \) markdown opportunities and deterministic demand can now be described as follows. Check first if \( \sum_{j=1}^{N} \mu_j y_j (\frac{1}{\hat{p}_0} + s_N) \leq I_1. \) If this inequality holds, then set \( p_0^n = \frac{1}{\hat{p}_0} + s_N \) and we are done. If not, then solve a single non-linear equation in one unknown, i.e., solve (16) after substituting from (17) into (16). The procedure can be easily implemented on a spreadsheet. In Fig. 1, we show how the optimal price

![Fig. 1. Optimal percent discount in period two of a two-period problem. Data: \( I_1 = 1000, \ s_2 = 0.1, \ K_1 = 1000, \ \beta_1 = 1, \ \mu_1 = 3, \ K_2 = 1000, \ \mu_2 = 4, \) and \( \beta_2 \) is varied from 1.1 to 11 in steps of 0.1.](image-url)
discount in the second period of a two-period problem depends on the ratio of $\beta_1$ and $\beta_2$ for a representative example. In agreement with intuition, the discount increases sharply as the mean reservation price declines, i.e., $\beta_1/\beta_2 \rightarrow 0$. We can also show (see corollary below) that the clearance prices are decreasing in the period index under declining reservation prices.

**Corollary 1.** If the mean reservation price is decreasing over time, i.e., $\beta_j^{-1} \geq \beta_{j+1}^{-1}$, for all $j \geq 1$, then the optimal prices are also decreasing, i.e., $p_j^0 \geq p_{j+1}^0$.

As proof of the above claim, observe first that it holds when $p_j^0 = \frac{1}{\beta_j} + s_N$. On the other hand, if we use Theorem 2 to find optimal prices, then Corollary 1 follows directly from Eq. (17).

We now consider the situation in which the mean reservation prices do not change over time, i.e., $\beta_j = \beta$ for all $j$, but we want to explore multiple opportunities to adjust prices. In this case, the optimal price has an explicit expression. If $\sum_{i=1}^N \mu_i y_i (1 + s_N) \leq I_1$, then the optimal prices are: $p_j = \frac{1}{p} + s_N$ for all $j$. If not, we then use Theorem 2 and it is easy to verify from (17) that $p_j = p_N$ for all $j = 1, 2, \ldots, N - 1$. Clearance price $p_N$ is obtained as follows: $p_N = \frac{1}{p} \left[ \frac{1}{\beta} \ln (1 + \sum_{j=1}^N \mu_j K_j) \right]$. Therefore, overall optimum clearance price is $p^b = \max \left\{ \frac{1}{\beta} + s_N \mid \ln (1 + \sum_{j=1}^N \mu_j K_j) \right\}$. The significance of this result is two-fold. First it shows that a single-discount pricing policy is optimal even when the demand arrival intensities $(\mu_j K_j)$ do vary from period to period. Put differently, we have shown that the mean reservation prices $\beta_j^{-1}$ are the primary determinant of relative prices in different periods. Second, it can be viewed as a multiperiod analog of Proposition 1 of Section 2. The result is also in agreement with a similar observation made by Gallego and Van Ryzin (1994).

It is sometimes convenient to have a single markdown and keep price unchanged in the remainder of the selling season, even if $\beta_j$'s vary from one period to the next. When additional markdowns incur the cost of relabeling merchandise with new price stickers, there are additional economic reasons for choosing a single-discount strategy. The methodology developed above allows us to quickly compute the magnitude of error that results from a single-discount scheme for any given example. The details are as follows. Let the price after exercising the opportunity to markdown once and the corresponding objective function be denoted with a subscript $f$. That is,

$$\pi^f_j(p_j) = p_j I_1 - (p_j - s_N) \left( I_1 - \sum_{i=1}^N d_i \right)^+.$$  

Note that (18) is similar to the single pricing opportunity case (Eq. (1)) with demand $\sum_{i=1}^N d_i$. Therefore, analogous to the analysis in Section 2, the optimal $p_j$ is either the unconstrained single-markdown price (if at that price total demand is still smaller than available inventory) or the stock-clearing price. The unconstrained single-markdown and the stock-clearing prices are obtained, respectively, by solving the following equations:

$$p : \sum_{i=1}^N K_i \mu_i e^{-\beta_p} [1 - \beta_j (p - s_N)] = 0.$$  

$$p : \sum_{i=1}^N K_i \mu_i e^{-\beta_p} = I_1.$$  

In Fig. 2, we plot the penalty for using the single-discount price $\{\pi^d(p^0) - \pi^f_j(p_j)\} \times 100/\pi^d(p^0)$, as a function of $\beta_1/\beta_2$ for the same data that is used to generate Fig. 1. Note that the penalty for using a single-markdown scheme declines both when the mean reservation prices are relatively constant over time ($\beta_1/\beta_2 \rightarrow 1$) and when the mean reservation price declines significantly in period 2 ($\beta_1/\beta_2 \rightarrow 0$), achieving its maximum for moderate decline. Retail managers are thus justified in using a single markdown, not only when customers’ reservation prices remain invariant over time, but also when they drop sharply after the first period.
How do optimal prices vary with $I_1$ and the demand arrival intensity parameters ($\mu$ and $K$)? We answer this question by considering two cases. If $\sum_{i=1}^{N} \mu_i y_i (i + s_N) \leq I_1$, then the optimal prices are $p_j = \frac{1}{\beta} + s_N$, which are unaffected by $I_1$, $\mu$ and $K$. On the other hand, if $\sum_{i=1}^{N} \mu_i y_i (i + s_N) > I_1$, then from (17) we notice that for each $n < N$ the amount by which $p_n$ exceeds $p_N$ depends only on the mean reservation prices. Thus, optimal prices in all periods are affected in exactly the same manner as the price in period $N$. If $I_1$ increases, $p_N$ needs to decrease in order to maintain the equality in Eq. (16). Similarly the left hand side of Eq. (16) is increasing in $\mu$ and $K$, implying that $p_N$ must increase in order to maintain the equality. Overall, it follows that prices are increasing in demand arrival intensity ($\mu$ and $K$) and decreasing in the amount of leftover stock at the beginning of the first period ($I_1$). These observations agree with intuition and with the results reported in Section 2.

3.2. N-Pricing opportunities with stochastic demand

It can be shown that the stochastic $N$-period clearance pricing problem does not have a myopic optimal solution even when demands in different periods are assumed independent (see Heyman and Sobel (1984, pp. 84–85) for conditions under which myopic solutions exist). That means that the problem of choosing optimal prices is a computationally difficult problem. What makes matters worse is that the demand for fashion items tends to be correlated across time. Without the benefit of a Markovian framework, pricing decisions across multiple periods would need to be jointly optimized. Smith and Achabal (1998) noted that the state space for this problem is extremely large.

What should a retail manager do? One alternative is to assume independence and determine prices using an algorithm from one of the articles we have discussed in Section 1. Another alternative is to develop a heuristic procedure that accounts for demand correlation. We explore the latter option in this section. Note that even though the independence assumption allows us to establish the structure of the solution, optimal prices often need to be computed using a heuristic algorithm anyway. Our procedure works as follows. We determine clearance prices for the remaining periods at the start of each period. When selecting these prices, we assume that the prices will not be updated in the remainder of the clearance season based on dynamic information about remaining inventory at the start of each period. However, in practice, only the next period’s price is implemented. The prices in remaining periods are updated on a rolling horizon basis with new
prices computed for the remaining periods at the start of each new period. Put differently, we solve a static version of the problem repeatedly to account both for demand correlation and the fact that prices can be adjusted dynamically.

Consistent with the approach outlined above, our efforts are focused first on deriving structural properties, bounds, and a heuristic solution procedure. Using Eq. (11), the $N$-period expected revenue can now be written as follows:

$$E[\pi^*(p)] = p_1 I_1 - E \left[ \sum_{n=1}^{N} (p_n - p_{n+1})(I_n - D_n)^+ \right],$$

(21)

where, as before, $p_{N+1}$ equals $s_N$, $I_n = (I_{n-1} - D_{n-1})^+$, and $D_n = y_n(p_n)\xi_n$. Note that we are using upper case $D_n$ to denote demand to underscore the fact that demand is now assumed random. For each price vector $p$, the components of demand vector $D$ may be arbitrarily correlated through the interdependence of the random components $\xi_n$. Also note that $(I_n - D_n)^+$ is zero if $I_1 < \sum_{i=1}^{n} D_i$ and $(I_1 - \sum_{i=1}^{n} D_i)$ otherwise. Therefore (21) can be further simplified to the following equivalent form:

$$E[\pi^*(p)] = p_1 I_1 - \sum_{n=1}^{N} (p_n - p_{n+1})E \left[ \left( I_1 - \sum_{i=1}^{n} D_i \right)^+ \right],$$

(22)

where our goal is to find a price vector $p^*$ that maximizes $E[\pi^*(p)]$. Owing to the complicated form of the right hand side of (22), it is difficult to prove concavity of $E[\pi^*(p)]$ in $p$. Thus, ensuring that a numerical solution identifies the optimal $p$ is a challenging problem.

Let $x = (x_1, \ldots, x_N)$ denote a realization of the random components of demand, i.e., $x_n$ is a realization of $\xi_n$ for each $n$. Let $v_n(p, x) = (I_1 - \sum_{i=1}^{n} y_i(p)xi)_+ = [g_n(p, x)]^+$, where $g_n(p, x) = I_1 - \sum_{i=1}^{n} y_i(p)x_i$. The function $g_n(p, x)$ is a linear function of $x$ and $[u]^+$ is a convex function of $u$. Thus, $v_n$ is the composition of a convex function with a linear function, which is known to be convex. Next, applying Jensen’s inequality, we see that $E[(I_1 - \sum_{i=1}^{n} y_i(p)\xi_i)^+] \geq (I_1 - \sum_{i=1}^{n} y_i(p)E[\xi_i])^+$. If we assume decreasing prices, then the following result is straightforward upon setting $d_i = y(p_i)E[\xi_i]$ in the equivalent deterministic problem.

**Proposition 4.** For each price vector $p$ such that $p_n \geq p_{n+1}$, $n = 1, 2, \ldots, N$, $E[\pi^*(p)] \leq \pi^d(p)$. Therefore, the deterministic problem can be used to obtain the following lower and upper bounds on the stochastic solution. Let $p^*$ be the optimal price vector for the stochastic demand problem within the set of all price vectors with declining prices, then

$$E[\pi^*(p^0)] \leq E[\pi^*(p^*)] \leq \pi^d(p^*) \leq \pi^d(p^0).$$

(23)

Since $p^*$ is unknown, $E[\pi^*(p^0)]$ and $\pi^d(p^0)$ are the lower and upper bounds (respectively) that we can compute relatively easily. The expected revenue $E[\pi^*(p^0)]$ can be estimated by sampling from the joint distribution of random components of demand and taking the average of realized values from (22). Note that the assumption of declining prices is not a serious limitation from a practical viewpoint since retailers behave in this way when setting prices for clearance merchandise. Numerical experiments reported in Section 4 show that $\pi^d(p^0)$ is a relatively loose upper bound. However, $E[\pi^*(p^0)]$ is quite tight. Thus $p^0$ is a good heuristic solution. However, we develop a heuristic solution which is even better, as explained below.

The framework necessary for obtaining the proposed heuristic solution is based on the generalized Jensen bounds discussed in Huang et al. (1977). In this method, the support $\Xi$ of the vector $\xi = (\xi_1, \ldots, \xi_N)$ is successively partitioned into finer divisions in order to improve the bound. In our problem setting, this method works as follows.

Let $k$ index the cells of a $v$-fold partition of $\Xi$. In the stochastic programming literature, cells are also called scenarios and we use these terms interchangeably. Cells of this partition are defined such that each
\( B_k \) is convex, \( B_k \neq \emptyset, B_k \cap B_k' = \emptyset \) if \( k \neq \ell \), and \( \bigcup_{k=1}^{\ell} B_k = \mathcal{S} \). The partition is denoted by \( \mathcal{B}^{(v)} \) where \( \mathcal{B}^{(v)} = \{ B_k, k = 1, \ldots, v \} \). Note that the following rectangular partition, which we use in our numerical experiments, has all of the above-mentioned properties:

\[
B_k = \left[ a_1^{(k)}, b_1^{(k)} \right] \times \left[ a_2^{(k)}, b_2^{(k)} \right] \times \cdots \times \left[ a_N^{(k)}, b_N^{(k)} \right],
\]

where \( N \) is again the number of periods. Let \( \gamma^k = P(B_k) \) be the probability on the cell, and \( \xi^k \) be the vector of conditional expectations of the random demand components on the cell. Specifically, \( \xi^k_i = \int_{B_k}(x_i | \gamma^k) dF_N(x) \), where \( F_N(x) \) is the joint CDF of \( \xi \). Having developed these partitions, and noting that \( v_p(p,x) \) is convex, we apply the generalized Jensen bounding method of Huang et al. (1977, p. 320) to obtain:

\[
E_{v_p}(p,x) \geq \sum_{k=1}^{v} \gamma^k v_p(x, \xi^k).
\]

Successively finer partitions, which can be obtained, for example, by dividing each cell into \( 2^N \) cells, lead to larger lower bounds in (25).

Substituting from (25) into (22), we have for each \( p \),

\[
E[\theta(p)] = p_1 I_1 - \sum_{n=1}^{N} (p_n - p_{n+1}) \sum_{k=1}^{v} \gamma^k \left( I_1 - \sum_{i=1}^{n} y_i(p_i) \xi^k_i \right)^+
\]

\[
\equiv \tilde{\pi}(p)
\]

\[
\leq p_1 I_1 - \sum_{n=1}^{N} (p_n - p_{n+1}) \left( I_1 - \sum_{i=1}^{n} y_i(p_i) \sum_{k=1}^{v} \gamma^k \xi^k_i \right)^+
\]

\[
= p_1 I_1 - \sum_{n=1}^{N} (p_n - p_{n+1}) \left( I_1 - \sum_{i=1}^{n} y_i(p_i) E \xi^k_i \right)^+
\]

\[
= \pi'(p).
\]

Inequality (28) follows from the fact that for all \( a, (a_1)^+ + \cdots + (a_N)^+ \geq (a_1 + \cdots + a_N)^+ \). Notice the upper bound becomes successively better as \( v \) increases and, in the limit, we obtain the true objective function of the stochastic problem corresponding to any price vector. In reality, the function \( \tilde{\pi}' \) stabilizes very quickly as seen in Fig. 3, which shows the plot of \( \tilde{\pi}'(p^k) \) as a function of \( v \). The problem data corresponds to a two-period problem in which the random component of demand has a bivariate normal distribution with a correlation coefficient of 0.5. Parameters of demand distribution are chosen such that the chance of a negative demand is almost zero. Negative demands (customer returns) are not uncommon when dealing with clearance merchandise, but they tend to be insignificant overall. Notice that when \( v \) exceeds about 50, there is little marginal gain from increasing the number of partitions.

From now on, we shall assume that \( v \) has been chosen appropriately so that the function \( E[\theta(p)] \approx \tilde{\pi}'(p) \) for all \( p \). Our efforts are therefore directed at optimizing the function \( \tilde{\pi}' \). The advantage of using the partitioning scheme is that it allows us to quickly compute \( \tilde{\pi}' \) for any given price vector, which is necessary for generating and refining heuristic solutions. For each price vector \( p \), let \( k_p^+(n) \) be the subset of cells/scenarios for which there is some inventory at the start of the \( n \)th period, i.e.,

\[
k_p^+(n) = \left\{ j : \sum_{i=1}^{n-1} y_i(p_i) \xi^j_i < I_1 \right\}.
\]

If \( k_p^+(n) \) is non-null, we further divide it into the following subsets:

\[
k_p^+(n) = \left\{ j : \sum_{i=1}^{n} y_i(p_i) \xi^j_i = I_1 \right\}.
\]
That is, \( k_p^>(n) \cup k_p^<(n) \) is the subset of scenarios under which all stock available at the beginning of period \( n \) is sold in that period.

**Proposition 5.** Price vectors for which \( k_p^>(n) \neq \emptyset \) but the set \( k_p^<(n) \cup k_p^<=(n) \) is empty, are dominated. This holds for any period index \( n = 1, \ldots, N \).

**Proof.** Let \( p \) be a price vector such that \( k_p^>(n) \neq \emptyset \) but the set \( k_p^<(n) \cup k_p^<=(n) \) is empty. This means that price in period \( n \) is chosen such that demand exceeds available inventory under each scenario. Note that prices in period \( n + 1, \ldots, N \) do not matter since there is nothing left to sell in those periods under any scenario. Consider what will happen if we increase \( p_n \) while keeping \( p_j, j \neq n \) fixed. It is easy to see that increasing \( p_n \) so long as \( \sum_{i=1}^n y_i(p_j) x_i^k \geq I_1 \) for all \( k \), and \( \sum_{j=1}^n y_j(p_j) x_j^k = I_1 \) for at least one \( k \), strictly increases expected revenue in period \( n \), while leaving revenue in other periods unchanged. Thus, for every price vector with the property mentioned above, it is easy to find an alternate price vector that yields higher overall expected revenue. Hence proved. \( \square \)

**Proposition 6.** Price vectors for which \( k_p^>(n) = \emptyset \) for at least one \( n < N \) are dominated.

**Proof.** The above result is the deterministic analog of Proposition 2. It implies a price vector that results in the sale of all inventory in \( n < N \) periods under all scenarios is dominated. The proof is realized by showing that for every price vector \( p \) that clears all stock in \( n < N \) periods under all scenarios, there exists another price vector \( p' \) with higher expected revenue. From Proposition 5, the set \( k_p^>(n) \cup k_p^<=(n) \) cannot be empty. Furthermore, if \( k_p^>(n) \) is non-empty, then we are done since there is at least one scenario under which we have something to sell in period \( n + 1 \). Thus, from now onward, we assume that only \( k_p^>(n) \) and \( k_p^<=(n) \) are non-empty.
In order to show that the price vector \( p \) is dominated, we now construct another price vector \( p' \) such that \( p_j = p'_j \) for all \( j \neq n, n + 1 \) and \( p'_{n+1} = p_n \). We also set \( p_n' \) just slightly more than \( p_n \). By making the increase small enough, we can ensure that whatever inventory is not sold in period \( n \) for scenarios in \( k_p(n) \), can be completely sold in period \( n + 1 \) and that demand under scenarios \( k_p(n) \) does not become smaller than available inventory at the start of period \( n \). This is always possible since the demand is strictly positive for all price in the range \([0, \infty)\) under all scenarios. More formally, \( p' \) is chosen such that

\[
\hat{\pi}'(p') = \sum_{i=1}^{n-1} \sum_{k=1}^{v} p_{i} \min \left\{ \left( I_1 - \sum_{r=1}^{i-1} y_r(p_r) \xi_{1}^k \right)^+, y_i(p_i) \xi_{i}^k \right\} \gamma^k + \sum_{k \in k_p(n)} \left( I_1 - \sum_{i=1}^{n-1} y_i(p_i) \xi_i^k \right) \gamma^k + \sum_{k \in k_p(n)} \left( p_n y_n(p_n) \xi_n^k + p_n \left( I_1 - \sum_{i=1}^{n} y_i(p_i) \xi_i^k \right) \right) \gamma^k > \hat{\pi}'(p).
\]

The last inequality follows from the fact that \( p_n > p_n \), \( p'_{n+1} = p_n \), and the sum of inventory sold in periods \( n \) and \( n + 1 \) is identical under the two price vectors. Hence proved. \( \Box \)

Propositions 5 and 6 imply that the optimal price vector is such that either there is inventory leftover under all scenarios at the end of period \( N \) (recall this is called the unconstrained solution), or prices are chosen such that \( \sum_{i=1}^{N} y_i(p_i) \xi_i^k = I_1 \) for at least one scenario \( k = 1, \ldots, v \). Finding the optimal price vector when \( \sum_{i=1}^{N} y_i(p_i) \xi_i^k = I_1 \) for some \( k \) continues to be a hard problem. We still need to consider the aggregate impact on revenues of a candidate price vector under all scenarios. However, if we focus only on scenario \( k \) for which \( \sum_{i=1}^{N} y_i(p_i) \xi_i^k = I_1 \), and ignore all other scenarios, we then recognize that this problem has the solution already obtained in Theorem 2. On the other hand, if there is ample stock, Eq. (22) reduces to \( E[\pi'(p)] = p_1 I_1 - \sum_{i=1}^{N} (p_n - p_{n+1}) (I_1 - \sum_{i=1}^{n} y_i(p_i) E(\xi_i)) \), and \( p_i = 1/\beta_i + s_N \) for all \( i \). Our overall heuristic approach can now be summarized as follows:

i. Set \( p_i = 1/\beta_i + s_N \) for all \( i \). If with these prices, \( \sum_{i=1}^{N} y_i(p_i) \xi_i^k \leq I_1 \) for all \( k \), then these are optimal prices and we are done.

ii. If step 1 does not yield a solution, set \( \sum_{i=1}^{N} y_i(p_i) \xi_i^k = I_1 \) and solve for the optimal price for each \( k = 1, \ldots, v \) using relationship (17). This will generate price vectors \( p^k \), \( k = 1, \ldots, v \), and as many possible values of \( \hat{\pi}' \). Choose the price vector that maximizes \( \hat{\pi}' \) as the heuristic solution, i.e., \( \hat{p}' = \arg \max_p \hat{\pi}'(p^k) \).

In order to test the accuracy of the heuristic suggested above, we obtain the optimal clearance prices and corresponding expected revenue via an exhaustive search. Computational effort associated with the search procedure can be greatly reduced by establishing upper and lower bounds on optimal prices, which are obtained as explained below.

Let \( p_n^* \) denote the optimal price in period \( n \). Then, it can be argued that the only reason to not choose \( p_n^* = 1/\beta_n + s_N \) is that there is insufficient inventory to meet the resulting demand under at least one scenario. Such inventory constraint restricts sales thereby providing the incentive to raise prices, or else properties established in either Proposition 5 or Proposition 6 no longer hold. This implies that \( p_n^* \geq 1/\beta_n + s_N \). The function \( \hat{\pi}'(p) \) can be seen as a convex combination of realized revenues under \( v \) scenarios, where the revenue under scenario \( k \) is as given in (12) after substituting \( g = \xi^k \). We know that prices in the deterministic problem are higher when demand scale is larger. Therefore, if we replace each \( \xi^k \) by \( \max_i \{ \xi^k \} \), for each \( k \), the solution to the resulting deterministic problem is an upper bound on the possible values of \( p_i \). Let this price vector be denoted by \( p^\text{max} \), then from above arguments we have established that \( 1/\beta_n + s_N \leq p_n^* \leq p^\text{max} \).
Analogous to the deterministic demand case, we can also use the model above to determine the optimal single markdown price. It is obtained as follows:

$$p^*_j = \arg \max_p \left\{ \max_{\xi} \left[ p I_1 - (p - s_N) \sum_{i=1}^N x(i) \right] \right\}.$$

Using arguments similar to those presented in the previous Section, it can be proved that if $I_1 > \sum_{i=1}^N y_i(\xi_i)$ for all $k$, then $p^*_j$ is obtained from solving Eq. (19). Since $p : \sum_{i=1}^N y_i(\xi_i) > I_1$ for all $k$ cannot be optimal, this leaves $N$ cases where in the $k$th case $\sum_{i=1}^N y_i(\xi_i) = I_1$. Each such case results in a non-linear equation in one unknown similar to Eq. (20). The optimal single discount price is the price that results in the overall maximum expected revenue.

4. Examples and insights

In this section, we report results of numerical experiments with two markdown opportunities (i.e., $N = 2$). We use the multivariate normal distribution to represent the random components of demand. Recall that our overall heuristic approach allows resetting of prices in the remaining periods at the start of each new period. However, in all of the results reported below, the prices are set only once. The results therefore provide examples of worst-case performance of the algorithm.

Fig. 4 shows the upper and lower bounds, the heuristic and optimal solutions, and the best single-discount price solution as a function of the ratio $\beta_1/\beta_2$. The two solid lines are the upper and lower bounds, i.e., $\pi^d(p^0)$ and $\hat{\pi}^d(p^0)$ respectively. The dashed line shows the optimal solution found via an exhaustive search, the dash-dotted line shows the heuristic solution, and the dotted line shows the solution obtained after using the optimal single discount price. In terms of our notation, the last three lines represent, respectively the quantities $\hat{\pi}^d(p^0)$, $\hat{\pi}^d(p^*)$, and $\hat{\pi}^d(p^*_j)$. We also calculated the performance of the bounds. As evident $\pi^d(p^0)$ is a loose upper bound and on average it is about 14.1% larger than the heuristic solution (maximum deviation 15.7% and minimum deviation 9.9%). The optimal solution and the heuristic solution are quite close with an average gap of 1.4% (maximum gap 1.9% and minimum gap 0.24%). In fact for $\beta_1/\beta_2$
greater than 0.4, the optimal and the heuristic solutions are indistinguishable. The penalty for using a single-discount pricing scheme varies from 0.064% to 4.56% of the heuristic solution, with an average error of 1.58%. The solid line in Fig. 5 shows the gap between $\pi'(p^0)$ and the heuristic solution, the dash-dotted line shows the difference between the optimal and the heuristic solution, and the dotted line shows the penalty due to a single-discount price scheme.

We also investigated the effect of demand correlation on the goodness of the heuristic solution. The results are plotted in Fig. 6. It shows the values of $\pi'(p^0)$ (top solid line), $\pi'(p^*)$ (dashed line), $\pi'(p^*)$, and $\pi'(p^0)$. Notice that the last two values are too close and appear as a single solid line at the bottom. The problem data has $I_1 = 1000$, $s_2 = 0.1$, $K_1 = K_2 = 1000$, $\beta_1 = 1$, $\mu_1 = 4$, $\sigma_1 = 1.25$, $\beta_2 = 2.0$, $\mu_2 = 3$, $\sigma_2 = 1.0$, and $\rho$ is varied from $-0.99$ to $0.96$ in steps of 0.1. The difference between the heuristic solution and the optimal solution is small with an average gap of about 0.5% (maximum 1.1%, minimum 0.41%). Increasing positive correlation tends to lower expected profits.
Two points are worth noting in Fig. 4. First, we see that the solution of the equivalent deterministic problem is a reasonable heuristic. It is easy to implement on a spreadsheet and fast solution can be obtained. Second, it appears that the single-discount pricing scheme underperforms the solution obtained from an equivalent deterministic problem. As a practical matter, this means that retail managers will do better to solve the (faster/easier) equivalent deterministic model with time-dependent reservation prices, than use a single markdown strategy. This is particularly true when reservation prices drop only moderately in later periods.

5. Impact of demand variability

Large retail chains operate stores in diverse markets for which they use different marketing and advertising strategies. The combination of native differences in the markets and differences in marketing campaigns creates situations in which two stores may be trying to sell comparable fashion items in two markets that have different joint distributions of demand. In such cases, it helps to know if there are certain characteristics of the market that make one store provably more profitable, given that pricing decisions are made optimally in each case. Such comparisons may also influence how a store’s performance is judged, how advertisement budgets are allocated, and the retailer’s future decisions concerning the choice of markets in which to operate stores.

In this Section we show that the expected revenue is ordered opposite to the convex ordering of partial sums of demands, regardless of the nature of demand correlation across time. Loosely speaking, this means that markets with smaller variability of demand are more profitable. The significance of this result is that it holds for arbitrary demand distributions in different periods that have an arbitrary correlation structure. In order to prove this result, we need the following additional notation and definitions.

If \( X \) and \( Y \) are arbitrary random variables, and \( E[f(X)] \leq E[f(Y)] \) for all convex functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) for which expectations exist, then \( X \leq_{\text{conv}} Y \) (see, for example, Shaked and Shanthikumar, 1994; p. 56). It follows that if \( X \leq_{\text{conv}} Y \), then \( E(X) = E(Y) \) and \( \text{Var}(X) \leq \text{Var}(Y) \) (Shaked and Shanthikumar, 1994; Chapter 2). Next, recalling that for each \( p_n \), \( u_n(p_n, x) \) is a convex function of \( x \), it is apparent that if \( \sum_{i=1}^{n} \xi_i \leq_{a} \sum_{i=1}^{n} \xi_i' \), then \( E[(I_1 - \sum_{i=1}^{n} D_i)^+] \leq E[(I_1 - \sum_{i=1}^{n} D_i')^+] \), when the two markets differ only in the random components of their respective demand functions. If we further assume declining prices, i.e., \( p_n \geq p_{n+1} \), for all \( n = 1, \ldots, N - 1 \), then this observation and (22) immediately lead to the Proposition below.

**Proposition 7.** For each declining price vector \( p \), if \( \sum_{i=1}^{n} \xi_i \leq_{a} \sum_{i=1}^{n} \xi_i' \) for each \( n \), then \( E[\pi^s(p)] \geq E[\pi^s'(p)] \).

Let \( p^* \) and \( p'^* \) be optimal price vectors when the demand vectors are \( D \) and \( D' \), respectively, where \( \sum_{i=1}^{n} \xi_i \leq_{a} \sum_{i=1}^{n} \xi_i' \) and all other parameters of the two demand distributions are identical. Then, Proposition 7 also implies that \( E[\pi^s(p^*)] \geq E[\pi^s'(p'^*)] \). That is, under the conditions outlined in Proposition 7, the market with demand vector \( D \) continues to be more profitable even when both markets charge their respective optimal prices.

Note that the convex ordering of partial sums is related to (but weaker than) the positive-linear-convex order (plcx in short), which in turn is weaker than the multivariate analog of the convex order defined above. (If \( X \) and \( Y \) are arbitrary \( n \)-dimensional random vectors, \( n \geq 1 \), and \( E[f(X)] \leq E[f(Y)] \) for all convex functions \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) for which expectations exist, then \( X \leq_{\text{plcx}} Y \) (see, for example, Shaked and Shanthikumar, 1994, Chapter 4).) Given two \( N \)-dimensional vectors \( X \) and \( Y \), we say that \( X \) is smaller than \( Y \) in the plcx order, denoted as \( X \leq_{\text{plcx}} Y \), if for all convex functions \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), \( E(f(a_1 X_1 + \cdots + a_N X_N)) \leq E(f(a_1 Y_1 + \cdots + a_N Y_N)) \) with \( a_i \geq 0 \), when the expectations exist. Thus, if \( \xi \leq_{\text{plcx}} \xi' \), then \( \sum_{i=1}^{n} D_i \leq_{\text{plcx}} \sum_{i=1}^{n} D_i' \) follows upon setting \( a = (y_1(p_1), \ldots, y_n(p_n), 0, \ldots, 0) \), where exactly first \( n \) components are positive and the rest are 0.
6. Summary and conclusions

Our discussions with managers at several major retailers have revealed that customers’ reservation prices decline over the selling season of fashion items and that demands are significantly correlated across time. It appears that such problems have not been addressed in the operations research literature. This paper presents both deterministic and stochastic models for single and multiple-markdown clearance pricing problems that have the features mentioned above.

The models provide several important managerial insights for the deterministic demand case. If the mean reservation prices do not change during the clearance period (in our notation, this means that \( b_j \approx b \)), then the problem essentially becomes a one-markdown problem and the optimal prices do not change during the clearance period. Similarly, if \( b_j \)’s change dramatically during the clearance period, e.g., \( b_1 \gg b_2 \), then the optimal approach is to attempt to clear nearly all of the inventory in the first period, and only first markdown price matters. In this case also, the practitioner may view the problem as essentially a one-markdown problem. However, if \( b_j \)’s change moderately during the clearance period (a realistic scenario), the optimal prices will not be constant over the clearance period and the penalty of applying a single markdown and keeping the price unchanged thereafter is significant.

Our numerical experiments suggest that it is generally better to apply a multiple-markdown deterministic model as a heuristic approach for determining clearance prices than a stochastic single-markdown model. This is good news due to the fact that the solution methods for the single and multiple-markdown stochastic models are complicated, whereas the solution method proposed here for the multiple-markdown deterministic problem is exact, efficient, and quite straightforward to implement in a spreadsheet program such as Microsoft Excel. The deterministic model is also less data-hungry because retailers need only estimate the average of the random component of demand in each period.

There are two results of theoretical interest presented in this paper. We obtain upper and lower bounds for the expected revenue and for the optimal prices for the stochastic multiple-markdown problem. We also show that when the joint distribution of demand is smaller in the convex order of partial sums of demand, the corresponding store can realize greater expected revenue.

Appendix A. Proof of Theorem 1

Upon differentiating (7), we obtain the first-order optimality equation as follows:

\[
\frac{dE[\pi'(p)]}{dp} = y(p)[\mu - \Theta(z)]\left[1 - \beta(p - s)\left(1 - \frac{z(1 - F(z))}{\mu - \Theta(z)}\right)\right] = y(p)[\mu - \Theta(z)]\left[1 - \frac{\beta(p - s)}{v(z)}\right] = 0.
\]  
(A.1)

Since \( z = 1/y(p) \), and \( \Theta(z) = F(\epsilon - z) \), it implies as \( p \to \infty, y(p) \to 0, \) and \( \Theta(z) \to 0 \). Therefore, the optimality Eq. (A.1) yields two roots: either \( p = \infty \), or \( p = p^* \) that satisfies the following equation:

\[
\frac{\beta(p - s)}{v(z)} = 1.
\]  
(A.2)

Furthermore, \( p^* \) is a unique solution in the range \( (s, \infty) \). This is seen by taking the derivative of the left-hand side (A.2) which yields:

\[
\frac{d[\beta(p - s)/v(z)]}{dp} = \beta \left(\frac{v(z) - (p - s)v'(z)}{v(z)^2}\right) > 0.
\]  
(A.3)
In the above inequality, which follows from (9), the prime notation signifies derivative with respect to \( p \). Recall that in (9) we proved \( v'(z) = (dv(z)/dz)(dz/dp) \leq 0 \), for all \( z \). Therefore, when \( p > s \), the left-hand side of (A.2) is monotone increasing in \( p \), resulting in a unique point at which the equality holds.

Using (7) and taking the limit as \( p \) approaches infinity, we observe that \( \lim_{p \to \infty} E[\pi'(p)] = sI \). However, for any \( p \geq s \), we know from the same expression that \( E[\pi'(p)] \geq sI \). The logic behind this simple on an intuitive level: since we can sell the any amount of surplus stock at salvage \( s \), any price greater than \( s \) should result in at least as much revenue as the lower bounding salvage-value based revenue. Technically, the proof follows from observing that \( \mu - \Theta(z) \geq 0 \) for all \( p \). Since \( p^* > s \), we see that the solution in (A.2) dominates the option to set an extremely high price. That is, we need only consider the solution in (A.2). It remains to show that the solution to (A.2) is indeed a maximum. In order to do that, we take the second derivative of (7) to obtain:

\[
\frac{\partial^2 E(\pi'(p))}{\partial p^2} \bigg|_{p=p^*} = \delta(p^*)[v'(z^*) - \beta] \leq 0,
\]

where we have used \( \delta(p) = [\mu - \Theta(z)]v(p)/v(z) \geq 0 \) for notational compactness. The above inequality also follows from the fact that \( v'(z) = (dv(z)/dz)(dz/dp) \leq 0 \), for all \( z \). This is true in particular at \( z = z^* \). The inequality (A.4) now results from noting that \( \delta(p^*) > 0 \) for all \( p \).

In summary, we have proved that \( p^* \) is a local maximum and that the objective function value at \( p^* \) is at least as much as the value at \( p = \infty \). Since there are only two roots of the first order optimality equation, \( p^* \) is indeed a unique maximum. \( \square \)

**Appendix B. Proof of Proposition 3**

Let \( p^k = (p^k_1, \ldots, p^k_N, \ldots, \cdot) \) denote a feasible price vector for the \( k \)th problem, where \( k < N \). Notice that prices in periods \( k + 1 \) through \( N \) can be set arbitrarily and have no impact on \( \pi^d_k \) and that \( p^j_k < \infty \) for each \( j \leq k \). In order to prove Proposition 3, we will show that for every \( p^k \), there exists a price vector \( p^{k+1} \) for the \((k + 1)\)th problem such that \( \pi^d_{k+1}(p^{k+1}) > \pi^d_k(p^k) \). Since this inequality holds for every \( k < N \), and in particular when \( p^k \) is set equal to \( p^k_I \), a repeated application of the inequality leads to the conclusion reported in Proposition 3.

Let \( d^r_j \) denote the demand in period \( j \) of the \( r \)th-period problem. Then, vectors \( p^k \) and \( p^{k+1} \) are feasible if and only if the corresponding demands satisfy the following equality: \( \sum_{j=1}^{k} d^r_j = \sum_{j=1}^{k+1} d^r_j = I_I \). We construct the vector \( p^{k+1} \) such that \( p^{k+1}_j = p^*_j \) for every \( j = 1, \ldots, k \), and \( p^{k+1}_{k+1} = p^k_{k+1} \). Prices in periods \( k + 2 \) through \( N \) do not matter. Notice that \( d^r_j = d^{k+1}_j \) for every \( j < k \), and that \( d^{k+1}_k \) is strictly positive, but no more than \( d^*_k \) (due to decreasing price sensitivity), for the chosen price \( p^{k+1}_{k+1} = p^k_{k+1} \). Now, we adjust the period-\( k \) price of the \((k + 1)\)th-period problem until the amount demanded in that period is exactly equal to \( d^*_k - d^{k+1}_k \). Since this quantity is strictly less than \( d^*_k \), it follows that \( p^{k+1} > p^k \). The total revenue in the \((k + 1)\)th-period problem is then:

\[
\pi^d_{k+1}(p^{k+1}) = \sum_{j=1}^{k+1} d^r_j p^{k+1}_j = \sum_{j=1}^{k} d^r_j p^*_j + p^{k+1}_k [d^*_k - d^{k+1}_k] + p^{k+1}_k d^{k+1}_k
\]

\[
= \sum_{j=1}^{k} d^r_j p^*_j + [p^{k+1}_k - p^*_k] [d^*_k - d^{k+1}_k] \geq \pi^d_k(p^k).
\]

The last inequality follows from the fact that \( p^{k+1}_k > p^*_k \), and \( d^*_k \geq d^{k+1}_k \). \( \square \)
Appendix C. Proof of Theorem 2

The following proof is presented for a $k$-period problem, for any $k$ such that $k = 1, \ldots, N$. The results reported in Theorem 2 are realized by setting $k = N$.

Let $L_k = p_k[1 - \sum_{n=1}^{k-1} |p_n - p_{n+1}|(I_1 - \sum_{i=1}^{n} d_i) + \sum_{n=1}^{k} \gamma_n |(\sum_{i=1}^{n} d_i) - I_1|]$, where multipliers $\gamma_n$ are non-negative for $n = 1, 2, \ldots, k - 1$ and $\gamma_k$ is unrestricted. Using standard Kuhn–Tucker conditions for non-linear optimization (see, for example, Luenberger, 1984, pp. 314–317) the optimal price vector must satisfy the following first order necessary conditions:

$$\frac{\partial L_k}{\partial p_j} = 0 \quad \text{for all } j = 1, 2, \ldots, k,$$

(C.1)

$$\gamma_n \left[ \left( \sum_{i=1}^{n} d_i \right) - I_1 \right] = 0, \quad \text{for all } n = 1, 2, \ldots, k, \text{ and}$$

(C.2)

$$\sum_{n=1}^{k} d_n = I_1.$$  

(C.3)

Thus, we have $2k$ simultaneous equations in as many unknowns. From their definition, we know that $\partial d_n / \partial p_n = -\beta_n d_n = -\beta_n K_n \mu_n e^{-\beta_n t}$. Using this relationship and differentiating $L_k$, we obtain the following equations:

$$\frac{\partial L_k}{\partial p_j} = d_j \left( 1 - \sum_{n=j}^{k-1} |p_n - p_{n+1}| \beta_j - \sum_{n=j}^{k} \gamma_n \beta_j \right),$$

(C.4)

where $j = 1, 2, 3, \ldots, k - 1$, and

$$\frac{\partial L_k}{\partial p_k} = \left( I_1 - \sum_{n=1}^{k-1} d_n \right) - \gamma_k \beta_k d_k.$$  

(C.5)

From the fact that $\sum_{n=1}^{k} d_n$ must be strictly less than $I_1$ for all $j = 1, 2, \ldots, k - 1$, and complimentary slackness conditions (see Eq. (C.2)) it follows that $\gamma_j = 0$ for all $j = 1, 2, \ldots, k - 1$. Substituting from Eq. (C.3) in Eq. (C.5) and setting the derivative to zero we obtain

$$(1 - \gamma_k \beta_k) d_k = 0,$$

(C.6)

from where it immediately follows that $\gamma_k = 1/\beta_k$. Recall that $d_k = K_k \mu_k e^{-\beta_k t}$ and it is strictly positive for all values of $p_k \in [0, \infty)$. Using arguments similar to those presented for the deterministic one-period problem it can be confirmed that setting $p_j = \infty$, for $j = 1, 2, \ldots, k$, results in a local minimum and hence we do not consider that possibility further.

Substituting the values of $\gamma_n$ in (C.4), and setting the derivatives to zero, we obtain the following first order necessary conditions, one for each $j = 1, 2, \ldots, k - 1$:

$$d_j \left( 1 - \sum_{n=j}^{k-1} |p_n - p_{n+1}| \beta_j - \beta_j / \beta_k \right) = 0.$$  

(C.7)

For $p_j \in [0, \infty)$, the above is equivalent to

$$\sum_{n=j}^{k-1} |p_n - p_{n+1}| \beta_j = 1 - \frac{\beta_j}{\beta_k},$$

(C.8)
which further simplifies to
\[ p_j = p_k + \frac{1}{\beta_j} - \frac{1}{\beta_k}. \]  
(C.9)

Substituting for \( p_j \)'s from above in (C.3), we obtain
\[ \sum_{j=1}^{k} K_j e^{-\beta_j \left[ p_j + \frac{1}{\beta_j} - \frac{1}{\beta_k} \right]} = I_1. \]  
(C.10)

It is easy to see that the left hand side of Eq. (C.10) is strictly decreasing in \( p_k \). Since, demand is at least as much as inventory at price \( p_k = s_N \), there is a unique solution of (C.10) such that \( p_k \in [s_N, \infty) \). That is, Eqs. (C.9) and (C.10) can be used to determine \( p_j \)'s uniquely for each \( j = 1, 2, \ldots, k \). Let this clearance price vector be denoted by \( p^0_k \).

The unique solution to the first order necessary conditions is a maximum (and therefore a global maximum) if \(-L_k\) is positive semidefinite at \( p^0_k \). To verify that we take second partial derivatives of \(-L_k\) and find that
\[ \frac{\partial^2 (-L_k)}{\partial p_j^2} = -d_j \left( 1 - \beta_j (p_j - p_k) - \sum_{n=j}^{k} \gamma_n \beta_j \right), \]  
(C.11)
\[ \frac{\partial^2 (-L_k)}{\partial p_j^2} = \beta_j d_j \left( 2 - \beta_j (p_j - p_k) - \sum_{n=j}^{k} \gamma_n \beta_j \right), \]  
(C.12)
\[ \frac{\partial^2 (-L_k)}{p_a p_b} = 0, \quad \text{for all } a \neq b, \text{ and } 1 \leq a, b \leq k - 1. \]  
(C.13)

Clearly, \( \frac{\partial^2 (-L_k)}{p_a p_b} \bigg|_{p^0_k} = \beta_j d_j \geq 0 \). It can now be verified that the determinants of the \( \ell \times \ell \) principal minor of the \((k-1) \times (k-1)\) Hessian matrix of the second derivatives of \(-L_k\) is positive for all \( \ell = 1, 2, \ldots, k-1 \). Therefore, \(-L_k\) is positive semi-definite at \( p^0_k \). Hence proved. \( \square \)

References


