

The Carbon-Constrained EOQ

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Abstract: In a recent paper, Benjaafar et al. (2010) consider a lot-sizing problem with a cap on carbon emissions. Based on numerical results, they suggest that it is possible, by making only adjustments in the order quantities, to significantly reduce carbon emissions without significantly increasing cost. In this paper, we provide analytical support for this numerical observation using the framework of the economic order quantity (EOQ) model. We provide a condition under which it is possible to reduce emissions by modifying order quantities. We also provide conditions under which the relative reduction in emissions is greater than the relative increase in cost and discuss factors that affect the difference in the magnitude of emission reduction and cost increase. We discuss the broader applicability of these results to other operational models.

Keywords: Inventory management; Economic order quantity; Carbon emissions; Sustainable operations

1. Introduction

This paper is motivated by observations made recently by Benjaafar et al. (2010) suggesting that it is possible to significantly reduce carbon emissions without significantly increasing cost by making only operational adjustments. Their observations were based on numerical results obtained for a lot sizing problem in which a firm decides on production/procurement quantities over a finite planning horizon consisting of discrete periods. The objective of the firm is to choose order quantities in each period so as to minimize the sum of its fixed ordering costs, inventory holding costs, and variable production/procurement costs, while satisfying a constraint on total emissions (the emission cap). Emissions are associated, similarly to cost, with initiating orders, holding inventory, and production/procurement. They observe that it is possible to modify order quantities to reduce emissions substantially and that in some cases this can be achieved without a substantial increase in cost. The observations made in (Benjaafar et al., 2010) are important because they point to an opportunity where operational flexibility could be leveraged to reduce emissions without resorting to costly investments in carbon-reducing technologies.

In this paper, we use the framework of the economic order quantity (EOQ) model to provide analytical support for these observations. We provide a condition under which it is possible to reduce emissions by modifying order quantities. We also provide conditions under which the relative reduction in emissions is greater than the relative increase in cost and describe when the difference between the two is maximized. We discuss the applicability of these results to other models and other operational settings. We show that, the key requirements are that the cost and emission functions yield different optimal solutions, implying that the cost tradeoffs are different from the emission tradeoffs, and that the cost function is *flat* around the optimal solution but can be steep elsewhere. We show that significant reductions in emissions can indeed be achieved without significant increases in cost whenever the flat region of the cost function coincides with the steep region of the emission function. We show that these features are present in other operational models, including the facility location and newsvendor models, among others.

The results in this paper indicate that the opportunity for reducing carbon emissions via operational adjustments exists whenever the operational drivers of emissions are different from the operational drivers of costs. In settings where this is not the case (e.g., operational decisions that reduce cost tend to also reduce emissions), operational adjustments will obviously be ineffective. In that case, investments in efforts that modify the emission function (e.g., investments in efforts or technologies that lead to reductions in the emission parameters of underlying processes and activities) would be necessary.

2. Problem Formulation and Results

Consider a firm that faces a constant demand with rate D per unit time. Each time the firm places an order (either with its internal production facilities or with an external supplier), it incurs a fixed cost A per order. The firm also incurs a holding cost h per unit kept in inventory per unit time, and a cost c per unit purchased or produced. Without loss of generality, we assume that orders are delivered with zero leadtime (a positive leadtime can be included and does not affect the solution to the problem); we also assume that the firm must satisfy all the demand (the analysis can be easily extended to settings with backorders). The objective of the firm is to choose an order quantity Q that minimizes its cost per unit time subject to the constraint on the amount of carbon emitted (this cap can reflect either government regulations imposed on the firm or a voluntary effort by the firm to reduce its emissions by a specified amount). Similar to cost, emissions are associated with ordering, production/purchasing, and inventory holding, with \hat{A} , \hat{h} and \hat{c} denoting the amount of carbon emissions associated per order initiated (e.g., due to transportation), per unit held in inventory per unit time (e.g., due to heating or refrigeration of stored inventory), and per unit purchased or produced (e.g., due to energy consumed in production), respectively. The amount of carbon emitted is constrained to be less than a certain cap C . The problem can then be formally stated as follows:

$$\text{Minimize } Z(Q) = \frac{AD}{Q} + \frac{hQ}{2} + cD \quad (1)$$

$$\text{subject to } \frac{\hat{A}D}{Q} + \frac{\hat{h}Q}{2} + \hat{c}D \leq C \quad (2)$$

Let \hat{Q}_{min} denote the order quantity that minimizes carbon emission (the *emission-optimal* solution), then it is easy to verify that $\hat{Q}_{min} = \sqrt{\frac{2\hat{A}D}{\hat{h}}}$ and the corresponding emission level is $E_{min} = \sqrt{2\hat{A}\hat{h}D} + \hat{c}D$. Consequently, the problem admits a feasible solution only if $C \geq E_{min}$. In the remainder, we assume that this condition is always satisfied. Also, let Q^* denote the order quantity that minimizes the total cost while ignoring the carbon emission constraint (the *cost-optimal* solution). Then, it is easy to see that $Q^* = \sqrt{\frac{2AD}{h}}$, which corresponds to the standard EOQ solution. The following theorem characterizes the optimal solution to (1)-(2).

Theorem 1 *Let*

$$Q_1 = \frac{\hat{C} - \sqrt{\hat{C}^2 - 2\hat{A}\hat{h}D}}{\hat{h}} \text{ and } Q_2 = \frac{\hat{C} + \sqrt{\hat{C}^2 - 2\hat{A}\hat{h}D}}{\hat{h}} \quad (3)$$

where $\hat{C} = C - \hat{c}D$. Then the optimal solution to problem (1)-(2) is

$$\hat{Q}^* = \begin{cases} Q^* & \text{if } Q_1 \leq Q^* \leq Q_2, \\ Q_1 & \text{if } Q^* \leq Q_1, \\ Q_2 & \text{if } Q^* \geq Q_2. \end{cases} \quad (4)$$

Furthermore, the emission level under the optimal order quantity is

$$E(\hat{Q}^*) = \begin{cases} E_{max} & \text{if } Q_1 \leq Q^* \leq Q_2 \\ C & \text{otherwise,} \end{cases} \quad (5)$$

where

$$E_{max} = \hat{A}\sqrt{\frac{hD}{2A}} + \hat{h}\sqrt{\frac{AD}{2h}} + \hat{c}D \quad (6)$$

and corresponds to the emission level in the absence of the carbon constraint (also corresponds to the emission level when the optimal order quantity is Q^*).

Proof: From constraint (2), we can show that the optimal order quantity must satisfy $Q_1 \leq \hat{Q}^* \leq Q_2$. If $Q_1 \leq Q^* \leq Q_2$, then obviously $\hat{Q}^* = Q^*$ and the corresponding emission is

$$E(Q^*) = \frac{\hat{A}D}{Q^*} + \frac{\hat{h}Q^*}{2} + \hat{c}D = \hat{A}\sqrt{\frac{hD}{2A}} + \hat{h}\sqrt{\frac{AD}{2h}} + \hat{c}D.$$

If $Q^* \leq Q_1$ then $\hat{Q}^* = Q_1$ because $Z(Q)$ is convex in Q and choosing a higher value for \hat{Q}^* will lead to a higher cost. Similarly, if $Q^* \geq Q_2$ then $\hat{Q}^* = Q_2$ because choosing a lower value for \hat{Q}^* will lead to higher cost. In both of these cases, constraint (2) is binding and, therefore, $E(\hat{Q}^*) = C$. \square

In the following proposition, we show that cost is indeed decreasing and convex in the emission cap C while emission is linearly increasing in C , implying that reducing the emission cap leads initially to a larger relative emission reduction than the relative cost increase (e.g., in the example illustrated in Figure 1, an emission reduction of 20% leads only to a 4% increase in cost).

Proposition 1 For $E_{min} \leq C \leq E_{max}$, emission is linearly increasing in C while emission is strictly decreasing and convex in C .

Proof: For emission, the result follows immediately from Theorem 1. For cost, consider first the case where $Q^* = \hat{Q}^*$. In this case, cost and emissions are unaffected by the cap C and the results holds. Next, consider the case where $Q_1 \geq Q^*$. In this case, we have

$$Z(\hat{Q}^*) = Z(Q_1) = \frac{AD\hat{h}}{\hat{C} - \sqrt{\hat{C}^2 - 2\hat{A}\hat{h}D}} + \frac{h(\hat{C} - \sqrt{\hat{C}^2 - 2\hat{A}\hat{h}D})}{2\hat{h}} + \hat{c}D.$$

Therefore,

$$\begin{aligned} \frac{\partial Z(\hat{Q}^*)}{\partial \hat{C}} &= \frac{\partial}{\partial \hat{C}} \left(\frac{AD\hat{h}}{\hat{C} - \sqrt{\hat{C}^2 - 2\hat{A}\hat{h}D}} + \frac{h(\hat{C} - \sqrt{\hat{C}^2 - 2\hat{A}\hat{h}D})}{2\hat{h}} \right) \\ &= \frac{\partial}{\partial \hat{C}} \left(\frac{(A\hat{h} + \hat{A}h)\hat{C} + (A\hat{h} - \hat{A}h)(\hat{C} - \sqrt{\hat{C}^2 - 2\hat{A}\hat{h}D})}{2\hat{A}\hat{h}} \right) \end{aligned}$$

$$= \frac{\hat{A}h + \hat{A}h}{2\hat{A}h} + \frac{\hat{A}h - \hat{A}h}{2\hat{A}h} \frac{\hat{C}}{\sqrt{\hat{C}^2 - 2\hat{A}hD}}.$$

Consequently,

$$\begin{aligned} \frac{\partial^2 Z(\hat{Q}^*)}{\partial \hat{C}^2} &= \frac{\hat{A}h - \hat{A}h}{2\hat{A}h} \frac{\sqrt{\hat{C}^2 - 2\hat{A}hD} - \frac{\hat{C}^2}{\sqrt{\hat{C}^2 - 2\hat{A}hD}}}{\sqrt{\hat{C}^2 - 2\hat{A}hD}} \\ &= \frac{\hat{A}h - \hat{A}h}{2\hat{A}h} \frac{-2\hat{A}hD}{(\sqrt{\hat{C}^2 - 2\hat{A}hD})^3} = \frac{(\hat{A}h - \hat{A}h)D}{(\sqrt{\hat{C}^2 - 2\hat{A}hD})^3}. \end{aligned}$$

Since $Q_1 \geq Q^*$ implies $\sqrt{\frac{2AD}{h}} \leq Q_1 \leq \sqrt{\frac{2\hat{A}D}{\hat{h}}}$, we have $\hat{A}h - \hat{A}h \geq 0$. Therefore, $\frac{\partial^2 Z(\hat{Q}^*)}{\partial \hat{C}^2} \geq 0$. In the remaining case of $Q_2 \leq Q^*$ we can show using similar arguments that $\frac{\partial^2 Z(\hat{Q}^*)}{\partial \hat{C}^2} \geq 0$. Consequently, the optimal cost function is convex with respect to the carbon cap C . \square

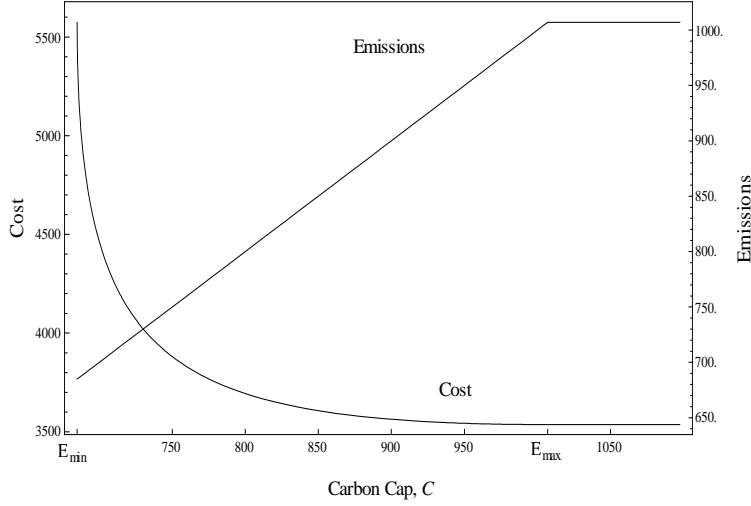
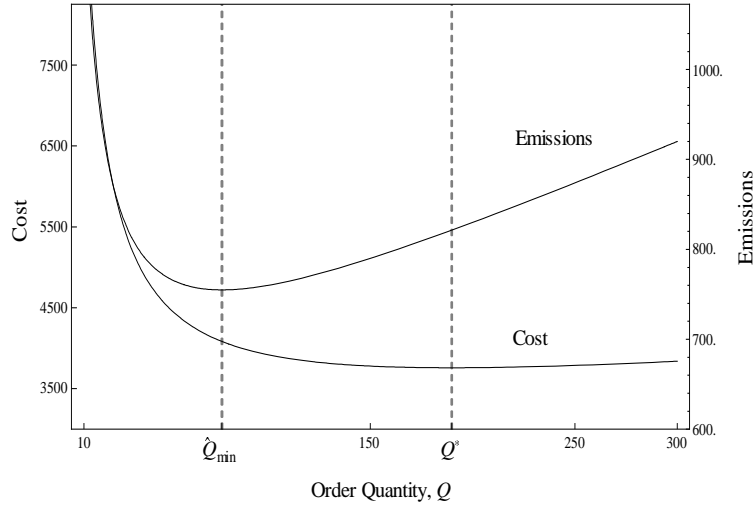


Figure 1: The impact of the carbon cap on emission and cost
 $(D = 600, A = 120, h = 2, c = 5, \hat{A} = 2, \hat{h} = 3, \hat{c} = 1)$

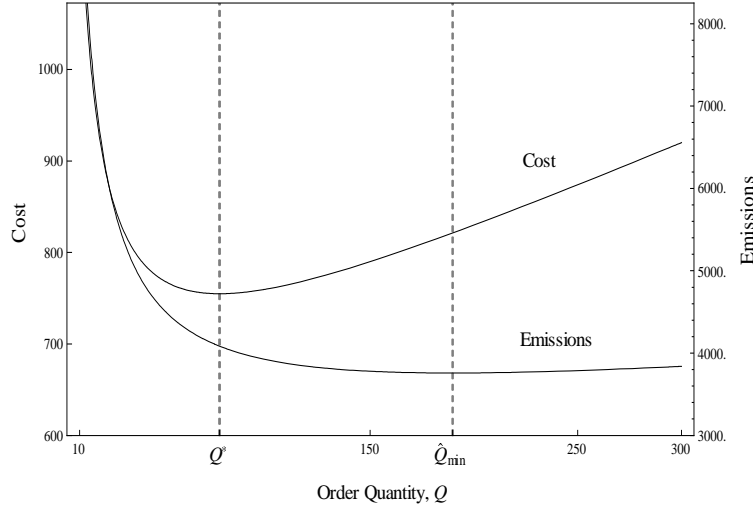
From Proposition 1, we can see that imposing an emission cap leads to an emission reduction only if the cap is sufficiently small, namely $C \leq E_{max}$. Otherwise, the cost-optimal solution is feasible and the corresponding emission is E_{max} . If $C \leq E_{max}$, emission is reduced by adjusting the order quantity (by either increasing it or decreasing it from the cost-optimal order quantity). However, for this to be possible, the cost-optimal order quantity must be different from the emission optimal solution. This leads to the following important corollary.

Corollary 1 *Reducing emissions by adjusting order quantities is possible only if $\frac{A}{h} \neq \frac{\hat{A}}{\hat{h}}$.*

This corollary follows from the fact that if $\frac{A}{h} = \frac{\hat{A}}{\hat{h}}$ then the cost-optimal solution is also emission-optimal (i.e., $Q^* = \hat{Q}_{min}$). In that case, emissions are already at their minimum and there is no operational adjustment that could further reduce them. On other hand, if $\frac{A}{h} \neq \frac{\hat{A}}{\hat{h}}$, there is an opportunity to reduce emissions by either increasing or decreasing the order quantity. In particular, if $\frac{A}{h} < \frac{\hat{A}}{\hat{h}}$ ($\frac{A}{h} > \frac{\hat{A}}{\hat{h}}$) then increasing (decreasing) the order quantity decreases emissions; see Figure 2 for a graphical illustration.



(a) ($D = 600, A = 120, h = 4, c = 5, \hat{A} = 10, \hat{h} = 2, \hat{c} = 1$)



(b) ($D = 600, A = 10, h = 2, c = 1, \hat{A} = 120, \hat{h} = 4, \hat{c} = 5$)

Figure 2: Cost versus Emissions

The broader implication of Corollary 1 is that operational adjustments can lead to emission reductions only if the cost parameters are not strongly correlated to the emission parameters, so

that what drives cost more is not what also drives emissions more. This is the case for example when fixed costs are higher than inventory holding costs but fixed emissions are lower than inventory-related emissions.

Although Corollary 1 identifies settings where it is possible to reduce emissions by adjusting order quantities, it does not specify the extent to which this reduction can be realized without significantly increasing cost. The examples shown in Figures 1 and 2 do suggest that indeed a modest reduction in the order quantity (away from the cost-optimal order quantity) leads to a modest increase in cost but a significant reduction in emission (both cost and emission have similar functional forms, with both being convex in Q and approaching ∞ as Q approaches either 0 or ∞). More importantly, the cost function for the EOQ is *flat* in the region around the optimal solution. This means that in this region a relative change in the order quantity leads to a lower relative increase in cost. In what follows, we characterize this flatness and identify a condition under which the reduction in emission is greater than the increase in cost (for further discussion and related results see Dobson (1987); Lowe and Schwarz (1983) and Zipkin (2000) for results and discussion).

Let $Z'(Q)$ and $E'(Q)$ refer to the components of cost and emission that are affected by order quantity. That is, $Z'(Q) = AD/Q + hQ/2$ and $E'(Q) = \hat{A}D/Q + \hat{h}Q/2$. Also, let,

$$\delta_Q = \frac{Q - Q^*}{Q^*}$$

denote the relative change in the order quantity (with respect to the cost-optimal order quantity), and

$$\delta_Z = \frac{Z'(Q) - Z'(Q^*)}{Z'(Q^*)} \text{ and } \delta_E = \frac{E'(Q^*) - E'(Q)}{E'(Q^*)}$$

denote respectively the corresponding relative change in cost and emission. Then, we can show that

$$\delta_Z = \frac{\delta_Q^2}{2(1 + \delta_Q)} \text{ and } \delta_E = -\frac{(1 - \alpha)\delta_Q + \delta_Q^2}{(1 + \alpha)(1 + \delta_Q)}, \quad (7)$$

where $\alpha = \frac{\hat{A}/\hat{h}}{A/h}$ is the ratio of the emission parameters to the cost parameters.

First, it is easy to see that even relatively large values of $|\delta_Q|$ lead to relatively small values of δ_Z . For example, increasing the order quantity by 30% ($\delta_Q = 0.3$) leads to an increase in cost of only 3.46% ($\delta_Z = 0.0346$); see Figure 3 for a full characterization of δ_Z as a function of δ_Q . Mathematically, we can show that

$$\frac{\partial \delta_Z}{\partial \delta_Q} = \frac{\delta_Q(2 + \delta_Q)}{2(\delta_Q + 1)^2},$$

which is equal to 0 for $\delta_Q = 0$. Moreover, we can show that if $\delta_Q > 0$, then $\delta_Z \leq \delta_Q$ (in other words, the relative increase in cost is always lower than the relative increase in the order quantity

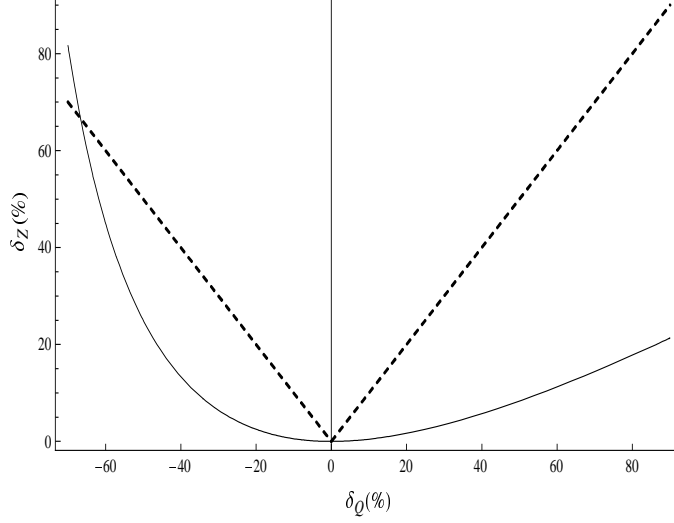


Figure 3: The impact of changes in order quantity on cost (Values of δ_Z above the 45 degree lines (shown in dashes) correspond to cases where δ_Z is smaller than δ_Q)

regardless of the size of the increase). If $\delta_Q < 0$, then $\delta_Z \leq \delta_Q$ as long as $\delta_Q \geq -2/3$ (i.e., we can reduce order quantity by as much as 2/3 with increasing cost by that much). In contrast,

$$\frac{\partial \delta_E}{\partial \delta_Q} = \frac{\alpha - (1 + \delta_Q)^2}{(1 + \alpha)(1 + \delta_Q)^2}$$

and takes on a value equal to $\frac{\alpha-1}{\alpha+1} \neq 0$ for $\alpha \neq 0$ when $\delta_Q = 0$. That is, while the cost function is always flat around Q^* , the emission function which can be quite steep (i.e., $|\frac{\alpha-1}{\alpha+1}|$ can be large). The fact that the flat region of the cost function coincides with the steep region of the emission function means that there is an opportunity to achieve more relative emission reductions than the corresponding relative increase in cost.

Next, we describe, the range over which order quantity can be adjusted while guaranteeing that the relative increase in cost is less than the relative reduction in emission, that is $\delta_Z \leq \delta_E$ and $\delta_E \geq 0$.

Proposition 2 For $\alpha > 1$, $\delta_Z \leq \delta_E$ if $0 \leq \delta_Q \leq \frac{2(\alpha-1)}{3+\alpha}$, and $\delta_Z \geq \delta_E$ otherwise. For $\alpha < 1$, $\delta_Z \leq \delta_E$ if $\frac{2(\alpha-1)}{3+\alpha} \leq \delta_Q \leq 0$, and $\delta_Z \geq \delta_E$ otherwise.

The proof immediately follows from the expressions of δ_Z and δ_E in Equation (7) and for brevity, we omit the details. As we can see, the interval over which the order quantity can be varied depends solely on α with its width increasing in the absolute value of the difference between $\frac{\hat{A}}{h}$ and $\frac{A}{h}$. In the limit cases of either $\alpha \rightarrow 0$ or $\alpha \rightarrow \infty$ the order quantity can be adjusted by as much as a factor

of 3. When $\delta_Z = \delta_E$ (and $\delta_Q = \frac{2(\alpha-1)}{3+\alpha}$) the resulting relative decrease in emission and in cost is given by

$$\delta_{E=Z} = \frac{2(\alpha-1)^3}{(1+3\alpha)(3+\alpha)} \quad (8)$$

which is also increasing in the absolute value of the difference between the ratios $\frac{\hat{A}}{h}$ and $\frac{A}{h}$. In the limit, as either $\alpha \rightarrow 0$ or $\alpha \rightarrow \infty$, $\delta_{E=Z} \rightarrow 2/3$, implying that emissions could be reduced by up to 2/3 without increasing cost by as much.

The following proposition further characterizes the tradeoff between cost and emission reductions.

Proposition 3 *Let $\delta_E(\delta_Z, \alpha)$ denote the relative emission reduction as function of the relative cost increase δ_Z and α . Then,*

$$\delta_E(\delta_Z, \alpha) = \begin{cases} -\frac{(\delta_Z + \sqrt{2\delta_Z + \delta_Z^2})(1 - \alpha + \delta_Z + \sqrt{2\delta_Z + \delta_Z^2})}{(1 + \alpha)(1 + \delta_Z + \sqrt{2\delta_Z + \delta_Z^2})}, & \text{if } \alpha > 1, \\ -\frac{(\delta_Z - \sqrt{2\delta_Z + \delta_Z^2})(1 - \alpha + \delta_Z - \sqrt{2\delta_Z + \delta_Z^2})}{(1 + \alpha)(1 + \delta_Z - \sqrt{2\delta_Z + \delta_Z^2})}, & \text{if } \alpha < 1. \end{cases} \quad (9)$$

Moreover,

- $\delta_E(\delta_Z, \alpha)$ is concave in δ_Z , it achieves its maximum value (the emission optimal solution) for $\delta_Z = \frac{(1-\sqrt{\alpha})^2}{2\sqrt{\alpha}}$ leading to an emission reduction $\delta_E = \frac{(1-\sqrt{\alpha})^2}{1+\alpha}$,
- for $\alpha > 1$, $\delta_E(\delta_Z, \alpha)$ is increasing concave in α (the same percentage increase in cost leads to a lower decrease on emission for a greater value of α), and
- for $\alpha < 1$, $\delta_E(\delta_Z, \alpha)$ is decreasing convex in α (the same percentage increase in cost leads to a lower decrease on emission for a lower value of α).

Proof: Expressing δ_Q as function of δ_Z leads to:

$$\delta_Q = \begin{cases} \delta_Z + \sqrt{2\delta_Z + \delta_Z^2} \geq 0, \\ \delta_Z - \sqrt{2\delta_Z + \delta_Z^2} \leq 0. \end{cases}$$

Substituting into the expression of δ_E leads to (9). It is easy to verify that

$$\frac{\partial \delta_E(\delta_Z, \alpha)}{\partial \alpha} = \begin{cases} \frac{2\sqrt{\delta_Z(\delta_Z+2)}}{(1+\alpha)^2} \geq 0, & \alpha > 1, \\ -\frac{2\sqrt{\delta_Z(\delta_Z+2)}}{(1+\alpha)^2} \leq 0, & \alpha < 1. \end{cases} \quad \text{and} \quad \frac{\partial^2 \delta_E(\delta_Z, \alpha)}{\partial \alpha^2} = \begin{cases} -\frac{4\sqrt{\delta_Z(\delta_Z+2)}}{(1+\alpha)^3} \leq 0, & \alpha > 1, \\ \frac{4\sqrt{\delta_Z(\delta_Z+2)}}{(1+\alpha)^3} \geq 0, & \alpha < 1. \end{cases}$$

Similarly,

$$\frac{\partial^2 \delta_E(\delta_Z, \alpha)}{\partial \delta_Z^2} = \begin{cases} \frac{(1-\alpha)}{(1+\alpha)(\delta_Z(\delta_Z+2))^{3/2}} \leq 0, & \alpha > 1, \\ \frac{(\alpha-1)}{(1+\alpha)(\delta_Z(\delta_Z+2))^{3/2}} \leq 0, & \alpha < 1. \end{cases}$$

Furthermore, solving for $\frac{\partial \delta_E(\delta_Z, \alpha)}{\partial \delta_Z} = 0$ leads to $\delta_Z = \frac{(1-\sqrt{\alpha})^2}{2\sqrt{\alpha}}$ and $\delta_E(\frac{(1-\sqrt{\alpha})^2}{2\sqrt{\alpha}}, \alpha) = \frac{(1-\sqrt{\alpha})^2}{1+\alpha}$, which completes the proof. \square

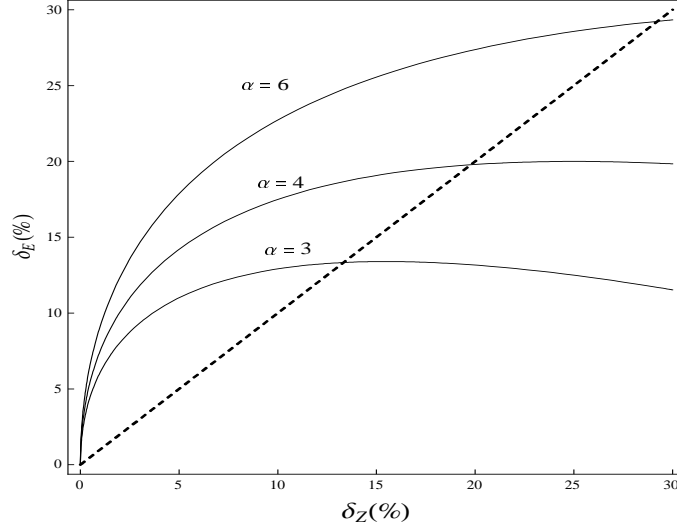


Figure 4: Cost-emission tradeoff

The cost-emission tradeoff is illustrated in Figure 4, where values of δ_E above the 45 degree lines (shown in dashes) correspond to cases where δ_E is larger than δ_Z . Figure 4 suggests that there is a value of δ_Z (and corresponding order quantity) which maximizes the difference $\delta_E - \delta_Z$. This value of δ_Z , to which we refer as δ_Z^{max} , is given by

$$\delta_Z^{max} = \frac{2(1+\alpha)}{\sqrt{(3+\alpha)(3\alpha+1)}} - 1, \text{ achieved for } \delta_Q^{max} = \sqrt{\frac{3\alpha+1}{3+\alpha}} - 1$$

The associated decrease in emission δ_E^{max} and the maximum difference ($\delta_E^{max} - \delta_Z^{max}$) are respectively given by

$$\delta_E^{max} = \frac{1}{1+\alpha} \left(1 - \sqrt{\frac{3+\alpha}{3\alpha+1}}\right) \left(\sqrt{\frac{3\alpha+1}{3+\alpha}} - \alpha\right), \text{ and}$$

$$\delta_E^{max} - \delta_Z^{max} = \frac{\sqrt{(3+\alpha)(3\alpha+1)}}{2(1+\alpha)} \left(1 - \sqrt{\frac{3+\alpha}{3\alpha+1}}\right) \left(\sqrt{\frac{3\alpha+1}{3+\alpha}} - 1\right).$$

These values can be viewed as maximizing the benefit derived from operational adjustments (the most environmental bang for the cost buck).

Proposition 4 $\delta_E^{max} - \delta_Z^{max}$ is increasing in $\alpha > 1$ and decreasing in $\alpha < 1$ and ranges from 0 to $(2 - \sqrt{3}) \approx 26,8\%$.

Proof: It is easy to show that

$$\frac{\partial(\delta_E^{max} - \delta_Z^{max})}{\partial\alpha} = \frac{2(\alpha - 1)}{(1 + \alpha)^2 \sqrt{(3 + \alpha)(3\alpha + 1)}} \begin{cases} > 0 & \text{for } \alpha > 1 \\ < 0 & \text{for } \alpha < 1 \end{cases}, \text{ and}$$

$$\lim_{\alpha \rightarrow 0}(\delta_E^{max} - \delta_Z^{max}) = \lim_{\alpha \rightarrow \infty}(\delta_E^{max} - \delta_Z^{max}) = 2 - \sqrt{3}. \quad \square$$

In the special cases of $\hat{A} = 0$ and $\hat{h} = 0$, $\delta_Z^{max} = \frac{2-\sqrt{3}}{\sqrt{3}} \approx 15.5\%$ and $\delta_E^{max} = \frac{\sqrt{3}-1}{\sqrt{3}} \approx 42.3\%$ (in other words, we achieve a 42.3% reduction emissions with only a 15.5% increase in cost).

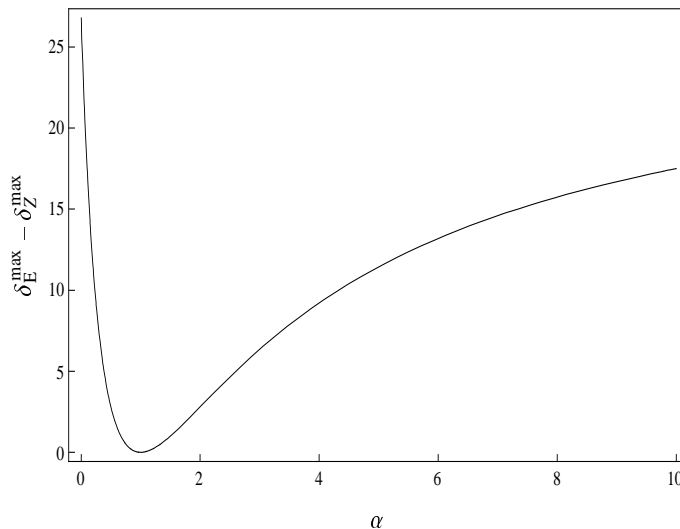


Figure 5: The impact of α on the maximum difference between δ_E and δ_Z

3. Extensions to Other Models

In the previous section, we showed how it may be possible to significantly reduce emissions without significantly increasing cost via only an operational adjustment. In particular, we showed that the key requirements are that the cost and the emission functions yield different optimal solutions and that the cost function exhibits less sensitivity around the cost-optimal solution than the emission function. These features can be shown to be present in several other operational settings and models. Therefore, the insights obtained have applicability to those cases as well. In what follows, we briefly describe two other widely used operational models for which this is the case.

3.1 The Facility Location Model

Consider the problem of determining the number of facilities N to serve demands that are uniformly distributed over a region with an area a and demand density ρ (amount of demand per unit of area). There is a fixed cost f incurred with each facility and a variable cost c incurred with each unit of demand transported one unit of distance. There is a tradeoff between the fixed costs and the transportation costs, with a higher number of facilities leading to higher fixed costs but, because the service area of each facility is a/N , lower transportation costs. Under the assumptions that a is a square area and travel occurs along paths oriented at 45 degrees to the sides of the square, the total expected cost $Z(N)$ can be shown to be given by (see for example Daganzo (1991) and Daskin (2008))¹

$$Z(N) = fN + c\rho a \frac{2}{3} \sqrt{\frac{a}{2N}}. \quad (10)$$

Similarly to cost, there is a fixed emission \hat{f} associated with each facility and variable emission \hat{c} associated with each unit of demand transported one unit of distance, leading to an expected total emission $E(N)$ given by

$$E(N) = \hat{f}N + \hat{c}\rho a \frac{2}{3} \sqrt{\frac{a}{2N}}. \quad (11)$$

We can then show that the cost-optimal number of facilities is given by

$$N^* = K \left(\frac{c}{f}\right)^{2/3}, \quad (12)$$

and the emission-optimal number of facilities by

$$\hat{N}^* = K \left(\frac{\hat{c}}{\hat{f}}\right)^{2/3}, \quad (13)$$

where $K = a \left(\frac{\rho}{2\sqrt{3}}\right)^{2/3}$.

As in the EOQ case, there is an opportunity to reduce emissions via an adjustment in the number of facilities whenever $\frac{\hat{c}}{\hat{f}} \neq \frac{c}{f}$. Moreover, we can show that the cost function is flat around the optimal solution and that whenever the flat region of the cost function coincides with the steep region of the emission function, a significant reduction in emissions can be achieved without a significant increase in cost. To see this let,

$$\alpha = \frac{\hat{c}}{\hat{f}} / \frac{c}{f}, \quad \delta_N = \frac{N - N^*}{N^*}, \quad \delta_Z = \frac{Z(N) - Z(N^*)}{Z(N^*)}, \quad \text{and} \quad \delta_E = \frac{E(N^*) - E(N)}{E(N^*)}.$$

Then, it is easy to verify that,

¹In general, $Z(N) = fN + \frac{c}{\sqrt{N}}g(a)$, where $g(a)$ is a function dependent on the shape of area a .

$$\delta_Z = \frac{1}{3}(\delta_N - 2 + \frac{2}{\sqrt{1 + \delta_N}}) \text{ and } \delta_E = \frac{1}{1 + 2\alpha}(1 + \delta_N + \frac{2\alpha}{\sqrt{1 + \delta_N}}).$$

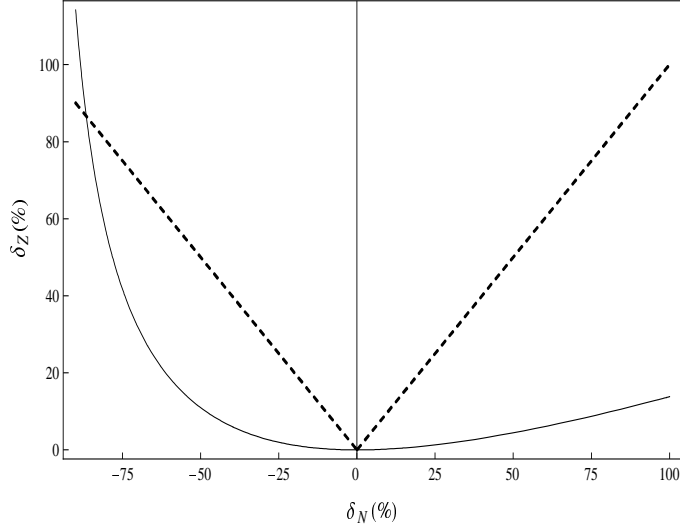


Figure 6: The impact of changes in the number of facilities on cost

The flatness of the cost function is apparent from Figure 6 (e.g., a 50% increase in N leads to less than 5% increase in cost). Mathematically, the cost ratio δ_Z is strictly convex in δ_N with $\frac{\partial \delta_Z}{\partial \delta_N}(0) = 0$. In contrast, the emission ratio δ_E is strictly concave in δ_N with $\frac{\partial \delta_E}{\partial \delta_N}(0) = \frac{\alpha-1}{1+2\alpha}$, which is increasing (decreasing) in $\alpha > 1$ ($\alpha < 1$) (recall that α is increasing in the absolute value of the difference $\frac{\hat{c}}{f} - \frac{c}{f}$). Moreover,

- $\delta_Z \leq \delta_E$, for $|\delta_N| \leq |\frac{1}{2}(\frac{9\alpha}{2+\alpha} - \sqrt{\frac{3(2+7\alpha)}{2+\alpha}})|$, and
- $\delta_E - \delta_Z$ is maximal for $\delta_N^{max} = (\frac{1+5\alpha}{4+2\alpha})^{2/3} - 1$.

In the limit case of $\alpha \rightarrow 0$, $\delta_E(\delta_N^{max}) - \delta_Z(\delta_N^{max}) \rightarrow 2 - 2^{2/3} \approx 0.412$ (we achieve approximately 60% reduction in emissions with only about 19% increase in cost) and in the limit case of $\alpha \rightarrow \infty$, $\delta_E - \delta_Z \rightarrow 2 - (5/2)^{2/3} \approx 0.15$ (we achieve approximately 26% reduction in emissions with only about 10% increase in cost).

3.2 The Newsvendor Model

Consider the problem of determining the order quantity Q for a single selling period given that demand for the period is stochastic and can be described by a continuous random variable D .

Ordering too little could lead to shortages while ordering too much could lead to leftover inventory, with a shortage cost c_s incurred per unit of demand that exceeds the quantity ordered Q and an overage cost c_o incurred per unit ordered that exceeds demand. Thus, the total expected cost for the period is given by

$$Z(Q) = c_s \mathbb{E}[(D - Q)^+] + c_o \mathbb{E}[(Q - D)^+], \quad (14)$$

where $\mathbb{E}[\cdot]$ refers to the expected value operator. Similarly, there are emissions associated with being short and with being over. In particular, \hat{c}_s is the amount of emissions per unit of shortage (e.g., emissions caused by resorting to fulfilling demand through more emission-intensive means) and \hat{c}_o is the amount of emissions per unit of overage (e.g., emissions associated with disposing of leftover inventory or with storing inventory until the next selling season). This leads to a total expected emission given by

$$E(Q) = \hat{c}_s \mathbb{E}[(D - Q)^+] + \hat{c}_o \mathbb{E}[(Q - D)^+]. \quad (15)$$

Then, it is easy to show that the cost-optimal order quantity Q^* is given by the solution to the following critical fractile equation (see for example Zipkin (2000)):

$$F(Q) = \frac{c_s}{c_s + c_o}, \quad (16)$$

and the emission-optimal order quantity \hat{Q}^* is given by the solution to

$$F(Q) = \frac{\hat{c}_s}{\hat{c}_s + \hat{c}_o}, \quad (17)$$

where F is the probability distribution of demand D .

As with the EOQ and the facility location models, there is an opportunity to reduce emissions by modifying the order quantity whenever $c_s/c_o \neq \hat{c}_s/\hat{c}_o$. For a variety of demand distributions, it is possible to show that the cost function is flat around the cost-optimal solution while the emission function is steep in the same region, leading to the possibility of significantly reducing emissions without significantly increasing cost.

Consider for example the case where demand is uniformly distributed over an interval $[a, b]$, with $b > a \geq 0$. We can easily verify that the cost-optimal order quantity is given by $Q^* = \frac{ac_o + bc_s}{c_o + c_s}$, the corresponding expected cost $Z(Q^*)$ and associated emission level $E(Q^*)$ are respectively given by

$$Z(Q^*) = \frac{c_s c_o}{2(c_s + c_o)}(b - a) \text{ and } E(Q^*) = (b - a) \frac{c_s^2 \hat{c}_o + c_o^2 \hat{c}_s}{2(c_s + c_o)^2}. \quad (18)$$

Let,

$$\delta_Q = \frac{Q - Q^*}{Q^*}, \delta_Z = \frac{Z(Q) - Z(Q^*)}{Z(Q^*)}, \text{ and } \delta_E = \frac{E(Q^*) - E(Q)}{E(Q^*)}.$$

Then, we can show that δ_Z is strictly convex in Q while δ_E is strictly concave in Q , with $\frac{\partial \delta_Z(Q)}{\partial Q}(0) = 0$, and $\frac{\partial \delta_E(Q)}{\partial Q}(0) \neq 0$.

Moreover, we can show that the maximum value δ_Q^0 by which the order quantity can be adjusted (increased (decreased) when $\alpha > 1$ ($\alpha < 1$)) while guaranteeing that $\delta_Z \leq \delta_E$ and $\delta_E \geq 0$ is given by

$$\delta_Q^0 = \left| \frac{4(a-b)(1-\alpha)}{\frac{c_o}{c_s} + 1(a + b\frac{c_s}{c_o})(1+\alpha)} \right|,$$

where $\alpha = \frac{\hat{c}_s/\hat{c}_o}{c_s/c_o}$. We can also show that δ_Q^{max} , the relative change in the order quantity for which the difference $(\delta_E - \delta_Z)$ is maximum, is given by

$$\delta_Q^{max} = \frac{2(a-b)(1-\alpha)}{\frac{c_o}{c_s} + 1(a + b\frac{c_s}{c_o})(1+\alpha)}.$$

When the order quantity is adjusted by δ_Q^{max} , we can easily verify that,

$$\delta_E(\delta_Q^{max}) = \left(2 + \frac{\alpha}{1 + \frac{C_o}{C_u}\alpha}\right) \delta_Z(\delta_Q^{max}). \quad (19)$$

This means that, independently of the parameters of the demand distribution, adjusting the order quantity by δ_Q^{max} leads to a reduction in emissions that is at least twice the increase in cost.

We conclude by noting that the results obtained in this paper are useful for settings other than those involving a strict emission cap. For example, firms may be subject to a tax on emissions or may be subject to a cap but are rewarded (penalized) for emitting less (more) than the cap as in a *cap-and-trade* system where rewards and penalties are determined by a prevailing market price. In the first case, we can show that a firm could significantly reduce its tax burden by making operational adjustments. In the second case, a firm could actually profit from reducing its emissions whenever the marginal reward from reducing emissions (e.g., carbon market price) is higher than the corresponding marginal cost associated with an operational adjustment.

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