

# Optimal Control of a Production-Inventory System with both Backorders and Lost Sales

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**Abstract:** We consider the optimal control of a production inventory-system with a single product and two customer classes where items are produced one unit at a time. Upon arrival, customer orders can be fulfilled from existing inventory, if there is any, backordered, or rejected. The two classes are differentiated by their backorder and lost sales costs. At each decision epoch, we must determine whether or not to produce an item and if so, whether to use this item to increase inventory or to reduce backlog. At each decision epoch, we must also determine whether or not to satisfy demand from a particular class (should one arise), backorder it, or reject it. In doing so, we must balance inventory holding costs against the costs of backordering and lost sales. We formulate the problem as a Markov decision process and use it to characterize the structure of the optimal policy. We show that the optimal policy can be described by three state-dependent thresholds: a *production base-stock level* and two *order-admission levels*, one for each class. The production base-stock level determines when production takes place and how to allocate items that are produced. This base-stock level also determines when orders from the class with the lower shortage costs (Class 2) are backordered and not fulfilled from inventory. The order-admission levels determine when orders should be rejected. We show that the threshold levels are monotonic (either nonincreasing or nondecreasing) in the backorder level of Class 2. We also characterize analytically the sensitivity of these thresholds to the various cost parameters. Using numerical results, we compare the performance of the optimal policy against several heuristics and show that those that do not allow for the possibility of both backordering and rejecting orders can perform poorly. © 2010 Wiley Periodicals, Inc. *Naval Research Logistics* 57: 252–265, 2010

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## 1. INTRODUCTION

Inventory problems treated in the literature fall mostly into two categories. One deals with systems where backordering is always allowed. The other deals with systems where backorders are not allowed and orders are considered *lost* if not fulfilled immediately. In practice, it is however common to have systems where both features, backorders and lost sales, are present. Backorders are allowed if the number of orders already backlogged is not too high but orders are rejected, or fulfilled through other means, otherwise. Such systems are consistent with human behavior since customers

are generally willing to wait but are not infinitely patient. Unfortunately, the analysis of systems with both backordering and lost sales is notoriously difficult. In fact, very little is known about optimal control policies or even effective heuristics for such systems. More significantly, little is known about the value of permitting both backorders and lost sales.

In this article, we address some of these limitations in the context of a production-inventory system with two classes of customers. In particular, we consider a continuous time and continuous review system where demand orders from each class arrive continuously over time one unit at a time with stochastic interarrival times. With each order arrival, a decision must be made on whether to fulfill the order from on-hand inventory, backorder it, or reject it. There are costs associated with the backordering and rejecting of orders, which can vary from class to class. Inventory is replenished from a production

Additional Supporting Information may be found in the online version of this article.

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facility that produces units one at time with stochastic production times. At any point in time, the system manager must decide on whether or not to produce and whether a produced item should be allocated to increase on-hand inventory or reduce backorders from one of the two classes, if there are any.

We formulate the problem as a Markov decision process (MDP) and characterize the structure of the optimal policy. We show that the optimal policy can be described by three state-dependent thresholds: a *production base-stock level* and two *order-admission levels*, one for each class. The production base-stock level determines when production takes place and how to allocate items that are produced. This base-stock level also serves as an inventory rationing threshold and determines when orders from the class with the lower shortage cost (Class 2) are backordered and not fulfilled from inventory. The order-admission levels determine when orders from each class are rejected. We show that all three threshold levels are monotonic (either nonincreasing or nondecreasing) in the backorder level of Class 2. We also show that the thresholds are monotonic in the various cost parameters.

Our results generalize several known results for simpler problems, such as those involving systems with only backorders or only lost sales. Our results can also be specialized to admission control problems in systems where inventory cannot be held (so-called make-to-order systems), leading to new results for that class of problems. Using numerical results, we compare the performance of the optimal policy against several heuristics and show that those that do not allow for the possibility of both backordering and rejecting or those that do not differentiate between customer classes can perform poorly. On the other hand, we find that a heuristic policy that mimics the optimal policy but uses fixed thresholds, instead of ones that are state-dependent, performs well over a wide range of parameters.

Our article is most closely related (in its theme) to the literature on inventory systems with *partial backordering*. A common assumption in that literature is that an arriving customer that faces a stockout is backordered with a certain probability and is lost otherwise (see, for example, Refs. [1–4], and the references therein). In situations where multiple orders are placed at once, this means that a fraction of customers are backordered while the remainder is lost. These probabilities or fractions are assumed to be exogenous and are supposed to reflect the willingness of customers to wait. Some authors have used a patience threshold to model customer's willingness to wait. For example, Posner [5] and Das [6] assume that customers are initially willing to wait, but if their demand is not fulfilled within their patience threshold time, they leave the system. In all of these article, partial backordering is an outcome of customer behavior and not a decision variable. A notable exception is Rabinowitz et al. [7] who assume that backorders are allowed but cannot exceed a certain fixed

threshold. With no exception, all the aforementioned articles assume a particular form for the inventory control policy, typically a base-stock or a  $(q, r)$  policy. Janakariam et al. [8] compare systems with pure backorder and pure lost sales and establish conditions under which the cost of one is higher than the other.

In addition to the literature on partial backordering, our article is related to the literature on inventory systems with multiple customer classes. Examples from this literature include the early work of Topkis [9], Nahmias and Demmy [10], and Cohen et al. [11] and the more recent work by Frank et al. [12], Deshpande et al. [13], Hu et al. [14], Zhao et al. [15], and the references therein. Articles that consider production-inventory systems, in the same way we do in this article, include Ha [16, 17], de Véricourt et al. [18], and Carr and Duenyas [19]. In each case, the authors show that there exists a threshold associated with each customer class such that it is optimal to fulfill demand from that class if on-hand inventory is above this threshold. Otherwise, depending on the assumption, demand from that class is backordered or lost. Most of this literature assumes either complete backordering or complete lost sales. There does not appear to be any article that consider both backorders and lost sales when there are multiple customer classes.

Finally, our article is more broadly related to the literature on admission control in queueing systems. A queueing system can be viewed as the make-to-order counterpart to the make-to-stock system we consider in this article. Reviews of this literature can be found in Stidham [20], Altman [21], and Koole [22]; for a recent example application see Savin et al. [23]. In Section 4.2, we show how results we obtain in this article can be applied to admission control problems to obtain new results regarding the structure of optimal policies in those settings.

Relative to the existing literature, our article makes the following contributions. It appears to be the first to characterize the structure of the optimal policy in systems where both backorders and lost sales are allowed and appears to be the first to do so in the context of systems with two customer classes. The results we present generalize several known results for simpler systems and apply to several other related problems, including admission control problems. Moreover, our article offers what appears to be the first evidence regarding the performance of several heuristics against the optimal policy and to assess the benefit of policies that permit both backordering and lost sales.

The rest of the article is organized as follows. In Section 2, we formulate the problem. In Section 3, we characterize the structure of the optimal policy. In Section 4, we discuss various special cases and related problems. In Section 5, we present results from a numerical study that evaluates the performance of the optimal policy against heuristics. In Section 6, we offer a summary and some concluding comments.

## 2. PROBLEM FORMULATION

We consider a system where a single product is produced at a single facility to fulfill demand from two customer classes. Items are produced one unit at a time with exponentially distributed production times with mean  $1/\mu$ . The production facility can produce ahead of demand in a make-to-stock fashion. However, items in inventory incur a holding cost  $h$  per unit per unit time. Customers place orders continuously over time according to a Poisson process with rate  $\lambda_i$  for customer class  $i$ ,  $i = 1, 2$ . Upon arrival, an order is either fulfilled from inventory, if any is available, backordered, or rejected. Orders from class  $i$  that are backordered incur a backordering cost  $b_i$  per unit per unit time, whereas orders that are rejected incur a lost sale cost  $c_i$  per unit, for  $i = 1, 2$ . We assume that penalties for shortage for Class 1 are greater than or equal to those of Class 2, so that  $b_1 \geq b_2$  and  $c_1 \geq c_2$ . This assumption is consistent with settings where one customer class is less tolerant of shortages (either delays or rejection) and, consequently, imposes higher penalties when they occur.

At any point in time, the system manager must decide whether or not to produce an item. If a unit is produced, a decision must also be made on whether to use it to increase inventory or to reduce backlog from one of the two classes, if there is any. We assume that preemption is possible, so that deciding not to produce could mean interrupting the production of a unit that was previously initiated. If interruption occurs, we assume it can be resumed the next time production is initiated (because of the memoryless property of the exponential distribution, resuming production from where it was interrupted is equivalent to initiating it from scratch). We assume that there are no costs associated with interrupting production. This conforms to earlier treatment of production-inventory systems in the literature; see, for example, Ha [16, 17]. This assumption is not restrictive since, as we show in Theorem 1, it turns out that it is never optimal to interrupt production of an item once it has been initiated.

At any point in time, the system manager must also decide on how to handle incoming orders. Should an order from class  $i$  arise, a decision must be made on whether to fulfill it from on-hand inventory, if there is any, to backorder it, or to reject it. For orders from Class 1, it is obvious that, if there is on-hand inventory, orders should be fulfilled immediately (we provide a rigorous proof for this in Section 3). However, for orders from Class 2, it may be advantageous to reserve on-hand inventory for future orders from Class 1, given the higher shortage penalties for Class 1. If an order, regardless of its class, is not fulfilled from inventory, then a decision must be made on whether it should be backordered or rejected. It may be more desirable to reject an order rather than backorder it if there is already a large backlog of orders from either class.

In our model, we assume that demand is Poisson and both production times and times between consecutive updates are

exponentially distributed. These assumptions are made in part for mathematical tractability as they allow us to formulate the control problem as an MDP and enable us to describe the structure of an optimal policy. They are also useful in approximating the behavior of systems where variability is high. The assumptions of Poisson demand and exponential production times are consistent with previous treatments of production-inventory systems; see for example, Buzacott and Shanthikumar [24], Ha [16, 17], Zipkin [25], and de Véricourt et al. [18], among others. In Section 5, we discuss how these assumptions may be partially relaxed.

The state of the system at time  $t$  can be described by the pair  $(X(t), Y(t))$ , where  $X(t)^+ = \max(0, X(t))$  denotes on-hand inventory,  $X(t)^- = \max(0, -X(t))$  denotes backorder level for customer Class 1, and  $Y(t)$  denotes backorder level for customer Class 2. Note that because of the possibility of interrupting production, it is not necessary to include in the state description whether an item is currently being produced or not. Furthermore, because both order interarrival times and production times are exponentially distributed, the system is memoryless and decision epochs can be restricted to only times when the state changes (i.e., the completion of an item or the fulfillment of an order). The memoryless property allows us to formulate the problem as an MDP and to restrict our attention to the class of Markovian policies for which actions taken at a particular decision epoch depend only on the current state of the system.

In each state, the system manager makes two types of decisions, one regarding production and the other regarding order fulfillment. For production, the choice is between not producing, producing to increase net inventory (either to increase on-hand inventory or decrease backorder level for Class 1), or producing to reduce backorder level from Class 2. For order fulfillment, the choice is between fulfilling an order, should one arise, from on-hand inventory, backordering the order, or rejecting it. A policy  $\pi$  specifies for each state  $\mathbf{x} = (x, y)$ , the action  $a^\pi(x, y) = (u_0, u_1, u_2)$ , where  $u_0 = 0$  means do not produce,  $u_0 = 1$  means produce to increase net inventory,  $u_0 = 2$  means produce to reduce backorders from Class 2,  $u_1 = 0$  means reject demand from Class 1,  $u_1 = 1$  means accept demand from Class 1 (either satisfy it from on-hand inventory if  $x > 0$  or backorder it otherwise),  $u_2 = 0$  means reject demand from Class 2,  $u_2 = 1$  means satisfy demand from Class 2 from on-hand inventory, and  $u_2 = 2$  means backorder demand from Class 2. For example, the action  $a^\pi(x, y) = (2, 1, 0)$  specifies that whenever the system is in state  $(x, y)$ , we should produce to reduce the backorder level for Class 2, accept orders from Class 1, and reject orders from Class 2.

Let  $N_i(t)$  denote the number of orders from class  $i$  that have been rejected up to time  $t$ . Then the expected discounted cost (the sum of inventory holding, backorder, and lost sales costs) over an infinite planning horizon  $v^\pi(\mathbf{x})$  obtained under a policy  $\pi$  and a starting state  $(x, y)$ , can be written as:

$$v^\pi(x, y) = E_{(x,y)}^\pi \left[ \int_0^\infty e^{-\alpha t} (hX^+(t) + b_1X^-(t) + b_2Y(t))dt + \sum_{i=1}^2 \int_0^\infty e^{-\alpha t} c_i dN_i(t) \right], \quad (1)$$

where  $\alpha > 0$ . (Extending the analysis to the case where the objective is to minimize average cost is straightforward and is briefly described at the end of Section 3). Our objective is to choose a policy  $\pi^*$  that minimizes the expected discounted cost. We refer to the optimal cost function as  $v^*$ , where  $v^* \equiv v^{\pi^*}$ .

Following Lippman [26], we work with a uniformized version of the problem in which the transition rate in each state under any action is  $\beta = \mu + \lambda_1 + \lambda_2$ , so that the transition times  $0 = t_0 \leq t_1 \leq t_2 \dots$  are such that the times between transitions  $\{t_{j+1} - t_j : j \geq 0\}$  form a sequence of i.i.d. exponential random variables, each with mean  $1/\beta$ . This leads to a Markov chain defined by  $\{Z_j : j \geq 0\}$  where  $Z_j = (X(t_j), Y(t_j))$  is the state resulting from the  $j$ -th transition. The introduction of the uniform transition rate allows us to transform the continuous time decision process into a discrete time decision process, simplifying the analysis considerably. To further simplify the analysis, and without loss of generality, we also rescale time by letting  $\alpha + \beta = 1$ . The optimal cost function  $v^*$  can now be shown to satisfy the following optimality equation:

$$v^*(x, y) = hx^+ + b_1x^- + b_2y + \mu T_0 v^*(x, y) + \lambda_1 T_1 v^*(x, y) + \lambda_2 T_2 v^*(x, y), \quad (2)$$

where the operators  $T_i, i = 0, 1, 2$ , are defined as follows:

$$T_0 v(x, y) = \begin{cases} \min\{v(x, y), v(x + 1, y), v(x, y - 1)\} & \text{if } y > 0 \\ \min\{v(x, y), v(x + 1, y)\} & \text{if } y = 0, \end{cases} \quad (3)$$

$$T_1 v(x, y) = \min\{v(x - 1, y), v(x, y) + c_1\}, \quad (4)$$

and

$$T_2 v(x, y) = \begin{cases} \min\{v(x - 1, y), v(x, y + 1), v(x, y) + c_2\} & \text{if } x > 0 \\ \min\{v(x, y + 1), v(x, y) + c_2\} & \text{if } x \leq 0. \end{cases} \quad (5)$$

Operator  $T_0$  is associated with the decision of whether or not to produce. If  $y = 0$  (there are no backorders from Class 2), then the choice is between not producing and producing to increase net inventory from  $x$  to  $x + 1$ . If  $y > 0$ , we must also consider the option of producing to reduce backorders for Class 2 from  $y$  to  $y - 1$ . Operator  $T_1$  is associated with

the decision of how to handle an order from Class 1. The choice is between rejecting the order and incurring the lost sale cost  $c_1$  or accepting the order (by either fulfilling it from on-hand inventory, if there is any, or backordering it). Accepting the order reduces net inventory from  $x$  to  $x - 1$ . Similarly, operator  $T_2$  is associated with the decision of how to handle orders from Class 2. The choice here is between rejecting the order which leaves the state unchanged, backordering it which increases backorder level from  $y$  to  $y + 1$ , or fulfilling (if there is on-hand inventory) which reduces net inventory from  $x$  to  $x - 1$ . Note that in determining the optimal action at a state  $(x, y)$ , the optimal cost function  $v^*$  is evaluated at each of the states to which the system would transition if any of the feasible actions is taken. For production, this means comparing  $v^*(x, y)$ ,  $v^*(x + 1, y)$ , and  $v^*(x, y - 1)$ , and choosing the action that corresponds to the lowest value. For order fulfillment for Class 1, this means comparing  $v^*(x, y) + c_1$  and  $v^*(x - 1, y)$ , whereas for orders from Class 2 the comparison is between  $v^*(x, y) + c_2$ ,  $v^*(x, y + 1)$ , and  $v^*(x - 1, y)$ .

### 3. THE STRUCTURE OF THE OPTIMAL POLICY

In this section, we characterize the structure of an optimal policy. To do so, we show that the optimal value function  $v^*(x, y)$  for all states  $(x, y)$  satisfies certain properties as specified in Definition 1 later. We then show that these properties imply a specific rule for the optimal action in each state.

**DEFINITION 1:** Let  $V$  be the set of functions defined on  $Z \times Z^+$ , where  $Z$  and  $Z^+$  are respectively the sets of integers and non-negative integers, such that if  $v \in V$ , then

- A1:  $v(x + 1, y) - v(x, y) \leq 0$  for  $x < 0$  and  $y \geq 0$  and  $v(x + 1, y) - v(x, y - 1) \leq 0$  for  $x < 0$  and  $y > 0$ ;
- A2:  $v(x, y - 1) - v(x, y) \leq 0$  for all  $x$  and  $y > 0$ ;
- A3:  $v(x + 2, y) - v(x + 1, y) \geq v(x + 1, y) - v(x, y)$  for all  $x$  and  $y$ ;
- A4:  $v(x, y + 2) - v(x, y + 1) \geq v(x, y + 1) - v(x, y)$  for all  $x$  and all  $y$ ;
- A5:  $v(x + 1, y + 1) - v(x, y + 1) \leq v(x + 1, y) - v(x, y)$  for all  $x$  and  $y$ ;
- A6:  $v(x + 2, y) - v(x + 1, y - 1) \geq v(x + 1, y) - v(x, y - 1)$  for all  $x$  and  $y > 0$ ;
- A7:  $v(x + 1, y + 1) - v(x, y) \geq v(x + 1, y) - v(x, y - 1)$  for all  $x$  and  $y > 0$ ; and
- A8:  $v(x, y) - v(x - 1, y) \geq -c_1$  for  $x > 0$  and all  $y$ .

**LEMMA 1:** If  $v \in V$ , then  $Tv \in V$ , where  $Tv(x, y) = hx^+ + b_1x^- + b_2y + \mu T_0 v(x, y) + \lambda_1 T_1 v(x, y) + \lambda_2 T_2 v(x, y)$ . Furthermore, the optimal cost function  $v^*$  is an element of  $V$ . That is,  $v^* \in V$ .

A proof of Lemma 1 and of all subsequent results can be found in the Appendix (Supporting Information). In the

proof, we first show that the operator  $T$  preserves properties A1–A8, which together with the convergence of value iteration, allows us to conclude that the optimal cost function  $v^*$  satisfies properties A1–A8. Applied to  $v^*$ , property A1 indicates that, whenever there are backorders from Class 1 ( $x < 0$ ), it is preferable to increase net inventory by one unit than to leave it unchanged. It is also preferable to increase net inventory by one unit than to reduce backorders from Class 2 by one unit (i.e., reducing backorders from Class 1 takes precedence over reducing backorders from Class 2). Property A2 indicates that, for a fixed level of net inventory, the fewer backorders from Class 2 the better. Property A3 implies that the marginal cost difference due to increasing net inventory (for a fixed level of backorders from Class 2) is nonincreasing. Similarly, A4 implies that the marginal cost difference due to increasing backorders from Class 2 (for a fixed level of net inventory) is nonincreasing. In other words, A3 and A4 indicate that the optimal cost function  $v^*$  is component-wise convex in  $x$  and  $y$ . Property A5 indicates that the marginal cost difference due to increasing net inventory is nonincreasing in the number of backorders from Class 2. In other words,  $v^*$  is submodular in the direction  $(x, y)$ . Property A6 indicates that the cost difference due to jointly increasing inventory and backorders by one unit each is nondecreasing in net inventory, whereas A7 indicates that the cost difference due to jointly increasing inventory and backorders by one unit each is nondecreasing in the backorder level. Property A8 implies that reducing on-hand inventory by one unit is preferable to leaving it unchanged and incurring the lost sale cost  $c_1$ . Some of the aforementioned properties, such as A5–A7, can be viewed as examples of properties of value functions involving generalized notions of modularity; see for example Koole [22, 27] and also Hajek [28]. However, it does not appear possible to use existing results, without additional analysis, to show that these properties are preserved when the optimality operator  $T$  is applied (in part because the minimization associated with operators  $T_0$  and  $T_2$  has more than two arguments).

To describe the optimal policy implied by the aforementioned properties, we define the following three threshold functions:

$$s^*(y) = \begin{cases} \min\{x|v^*(x+1, y) - v^*(x, y-1) > 0\} & \text{if } y > 0 \\ \min\{x|v^*(x+1, y) - v^*(x, y) > 0\} & \text{if } y = 0, \end{cases} \quad (6)$$

$$w_1^*(y) = \min\{x|v^*(x, y) - v^*(x-1, y) \geq -c_1\}, \quad (7)$$

and

$$w_2^*(y) = \begin{cases} \min\{x|v^*(x, y) - \min\{v^*(x-1, y), v^*(x, y+1)\} \geq -c_2\} & \text{if } x > 0 \\ \min\{x|v^*(x, y) - v^*(x, y+1) \geq -c_2\} & \text{otherwise.} \end{cases} \quad (8)$$

We are now ready to state our main result.

**THEOREM 1:** There exists an optimal stationary policy that can be specified in terms of a state-dependent production base-stock level  $s^*(y)$  and two state-dependent order admission levels  $w_1^*(y)$ , and  $w_2^*(y)$  as follows:

Optimal production policy

- (P1) Produce to increase on-hand inventory if  $y = 0$  and  $0 \leq x < s^*(0)$  or if  $y > 0$  and  $0 \leq x < s^*(y)$ .
- (P2) Do not produce if  $y = 0$  and  $x \geq s^*(0)$ .
- (P3) Produce to fulfill backorders from Class 2 if  $y > 0$  and  $x \geq s^*(y)$ .
- (P4) Produce to fulfill backorders from Class 1 if  $x < 0$ .
- (P5) It is never optimal to interrupt production once it has been initiated.

Optimal order fulfillment policy for Class 1

- (P6) Fulfill orders from Class 1 from on-hand inventory if  $x > 0$ .
- (P7) Backorder orders from Class 1 if  $w_1^*(y) < x \leq 0$ .
- (P8) Reject orders from Class 1 if  $x \leq w_1^*(y)$ .

Optimal order fulfillment policy for Class 2

- (P9) Fulfill orders from Class 2 from on-hand inventory if  $x > s^*(y+1)$ .
- (P10) Backorder orders from Class 2 if  $w_2^*(y) < x \leq s^*(y+1)$ .
- (P11) Reject orders from Class 2 if  $x \leq w_2^*(y)$ .

Moreover, the base-stock and admission levels have the following properties:

- (P12)  $s^*(y)$  is nonincreasing in  $y$ ,
- (P13)  $w_1^*(y)$  is nondecreasing in  $y$ ,
- (P14)  $w_2^*(y)$  is nondecreasing in  $y$ , and
- (P15)  $s^*(y) \geq 0$  and  $w_1^*(y) \leq 0$ .

The optimal policy is illustrated for an example system ( $\mu = 0.50, \lambda_1 = 0.36, \lambda_2 = 0.21, c_1 = 357.00, c_2 = 186.00, b_1 = 4.80, b_2 = 0.50, h = 1.50$ ) in Figs. 1–4. As we can see, the optimal policy defines several regions in the state space where different combinations of order fulfillment and production decisions are optimal. The contours of these regions are determined by the threshold functions  $s^*(y)$ ,  $w_1^*(y)$ , and  $w_2^*(y)$ . Note that only a subset of the state space is recurrent, and once the system enters the recurrent region it never leaves it. Within the recurrent region the amount of on-hand inventory is always bounded by  $s^*(0)$  and the amount of backorders from Class 1 is always bounded by  $-w_1^*(0)$ . That is, we always have  $w_1^*(0) \leq x \leq s^*(0)$ .

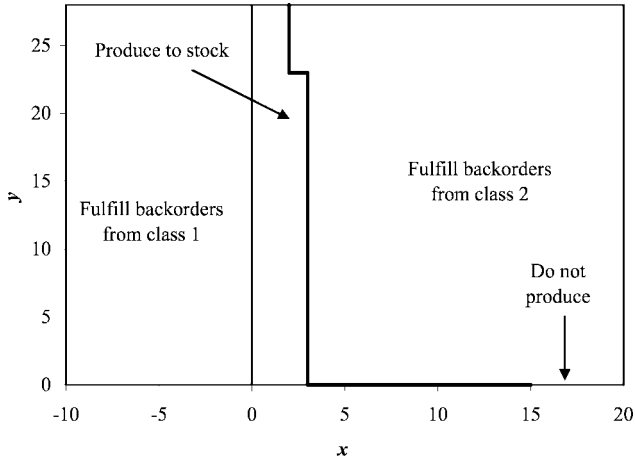


Figure 1. The optimal production policy.

Similarly, the amount of backorders from Class 2 is always bounded in the recurrent region. The maximum number of backorders of Class 2 that may be allowed is given by  $y_{\max} = \max\{y : s^*(y) = w_2^*(y)\}$ . The recurrent region for the example system is shown in Fig. 4.

In addition to placing limits on inventory and backorders, the optimal policy determines how inventory is allocated among the two classes. The base-stock level  $s^*(y)$  can be viewed as an inventory rationing parameter. If net inventory is below  $s^*(y)$ , then it is optimal to reserve inventory for future orders from Class 1 and to either reject or backorder orders from Class 2. Similar rationing applies to production, with priority given to increasing on-hand inventory over satisfying backorders from Class 2 if net inventory is below  $s^*(y + 1)$ .

In specifying the optimal policy, we chose in Theorem 1 to define the admission level for Class 2 in terms of a threshold  $w_2^*(y)$  on net inventory  $x$ . We could have equivalently

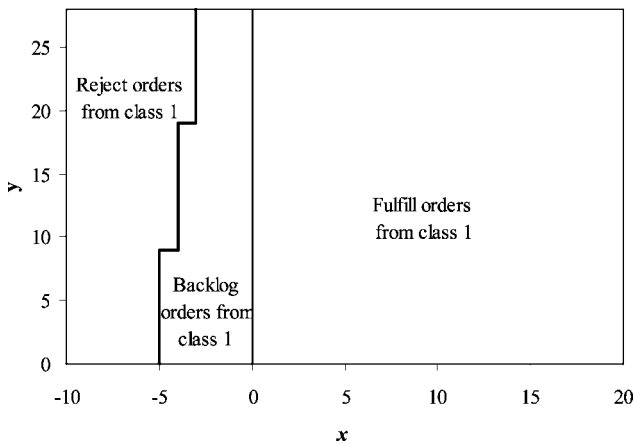


Figure 2. The optimal order fulfillment policy for Class 1.

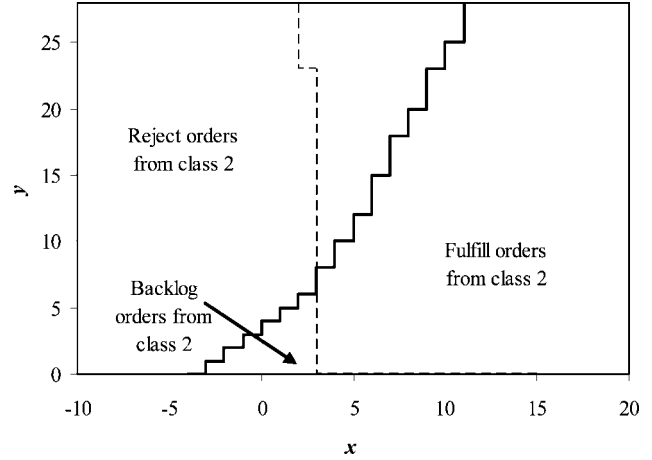


Figure 3. The optimal order fulfillment policy for Class 2.

chosen to define the admission level in terms of a threshold on the number of backorders for Class 2 (there exists a threshold  $r^*(x)$  such that it is optimal to reject orders from Class 2 if  $y \geq r^*(x)$  and to accept these orders otherwise). The admission level  $r^*(x)$  is nondecreasing in  $x$ .

In the following theorem, we characterize the impact of various cost parameters on the base-stock level and the admission thresholds.

**THEOREM 2:** The optimal base-stock level and the admission thresholds satisfy the following properties.

- (Q1) The base-stock level  $s^*(y)$ , for  $y \geq 0$ , is nonincreasing in the holding cost  $h$  and nondecreasing in the backorder cost  $b_1$  and the lost sales costs  $c_1$  and  $c_2$ .
- (Q2) The base-stock level  $s^*(y)$ , for  $y > 0$ , is nonincreasing in the backorder cost  $b_2$ .

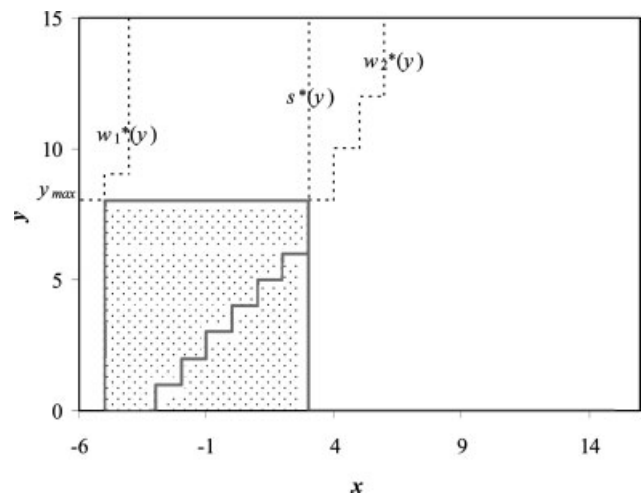
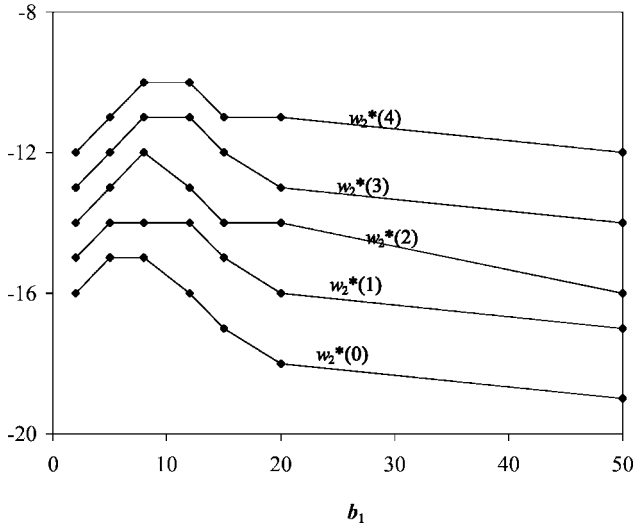


Figure 4. The recurrent region under the optimal policy.



**Figure 5.** The effect backorder cost  $b_1$  on the admission threshold  $w_2(y)$  ( $\mu = 1, \lambda_1 = 0.45, \lambda_2 = 0.45, c_1 = 500, c_2 = 250, h = 0.15, b_2 = 1.5$ ).

- (Q3) The admission threshold for class  $i$   $w_i^*(y)$ , for  $y \geq 0$ , is nondecreasing in the backorder cost  $b_i$  and nonincreasing in the lost sales cost  $c_i$  for  $i = 1, 2$ .
- (Q4) The admission threshold for class  $i$ ,  $w_i^*(y)$ , for  $y \geq 0$ , is nonincreasing in the holding cost  $h$ , for  $i = 1, 2$ .

Property Q1 indicates that it is optimal to stock less inventory in anticipation of future demand as the holding cost increases but to stock more as the backorder cost of Class 1 increases or the lost sales costs of Classes 1 and 2 increase. Property Q2 shows that higher priority, in both production and inventory allocation, is given to Class 2 as the backorder cost of Class 2 increases. Q3 indicates that it is optimal to backorder less from a class (higher admission threshold) as the backorder cost for that class increases but to backorder more (lower admission threshold) as the lost sales cost from that class increases. Q4 shows that it is optimal to backorder more as the holding cost increases.

The aforementioned monotonicity results do not appear to extend to the impact of shortage costs (either backorder or lost sales) of one class on the admission thresholds of the other class. For example, we observed numerically that increasing the backorder cost of Class 1 could either increase or decrease the admission threshold  $w_2^*(y)$  of Class 2; see Fig. 5. This perhaps surprising result is due to the complex interactions between the cost parameters and the various thresholds. Depending on how much an increase in  $b_1$  increases the base-stock level  $s^*(y)$  and the admission threshold  $w_1^*(y)$  this could make it more or less advantageous to admit more or less orders from Class 2. This effect is further compounded by the integrality of the various thresholds.

The cost parameters can also be used to characterize conditions under which the optimal policy further simplifies. For example, when the backorder cost from a particular customer class is sufficiently high, it becomes optimal to always reject orders from that class when there is no inventory on-hand. A sufficient condition for this to occur is  $b_i \geq c_i(\mu + \alpha)$  for  $i = 1, 2$ . The condition follows from comparing the cost of rejecting an order from class  $i$ ,  $c_i$ , to the expected (discounted) cost of backordering the order but immediately initiating the production of an item to fulfill it,  $b_i/(\mu + \alpha)$ . Similarly, when the lost sales cost of a class is sufficiently high, it becomes optimal never to reject an order from that class and to always backorder instead. A sufficient condition for this to occur is  $c_i \geq b_i/\alpha$  for  $i = 1, 2$ . The condition follows from comparing the cost of rejecting an order from class  $i$ ,  $c_i$ , to the expected cost of backordering it and having it stay in backorder forever,  $b_i/\alpha$ . We can also obtain a condition on when it is optimal to never hold inventory and to produce instead in a *make-to-order* fashion. This would of course occur when the holding cost is sufficiently high. A sufficient condition is  $h/(\lambda_1 + \lambda_2 + \alpha) \geq \max(b_i/\alpha, c_i)$  for  $i = 1, 2$ . The condition follows from comparing the expected cost of holding a unit in inventory from the time it completes production until the first next order arrival takes place,  $h/(\lambda_1 + \lambda_2 + \alpha)$ , to the maximum of the two costs of either rejecting the order,  $c_i$ , or backordering and having stay in backorder forever,  $b_i/\alpha$ . We should note that all these conditions are sufficient but not necessary, so it might be possible to identify other more restrictive conditions that imply each of the above special cases.

We conclude this section by noting that our analysis can be extended to the case where the optimization criterion is the average cost per unit time instead of the expected discounted cost. Given a policy  $\pi$ , the average-cost is given by:

$$J^\pi(x, y) = \limsup_{T \rightarrow \infty} \frac{1}{T} E_{(x,y)}^\pi \left\{ \int_0^T e^{-\alpha t} (hX^+(t) + b_1X^-(t) + b_2Y(t)) dt + \sum_{i=1}^2 \int_0^T e^{-\alpha t} c_i dN_i(t) \right\}. \quad (9)$$

A policy  $\pi^*$  that yields  $J^*(x, y) = \inf_\pi J^\pi(x, y)$  for all states  $(x, y)$  is said to be optimal for the average cost criterion. In the following theorem, we show that the optimal policy retains all of the properties observed in Theorem 1 under the expected discounted cost criterion.

**THEOREM 3:** The optimal policy under the average cost criterion retains all the properties of the optimal policy under the discounted cost criterion, namely that the production policy is determined by a state-dependent base-stock level and order fulfillment is determined by state-dependent admission

levels for each class. The base-stock and admission levels satisfy properties P1–P15 specified in Theorem 1. Furthermore, the optimal average cost is finite and independent of the initial state; that is, there exists a finite constant  $J^*$  such that  $J^*(x, y) = J^*$  for all states  $(x, y)$ .

#### 4. SPECIAL CASES AND RELATED SYSTEMS

There are several problems of interest which can be viewed as special cases of the problem we have considered so far. Some of these special cases have been treated previously in the literature. For some of these cases, the structure of the optimal policy can be further specified. In this section, we describe few of these cases and the additional structural results that can be obtained. In Section 5, we use some of these cases as a basis for constructing heuristic policies.

##### 4.1. Systems with a Single Customer Class

An important special case is a system with a single customer class. Let  $\lambda$  be the arrival rate,  $b$  the backorder cost per unit per unit time, and  $c$  the lost sale cost per unit. In this case, the state space can be described in terms of net inventory only and the optimality equation (under the discounted cost criterion) simplifies to:

$$v^*(x) = hx^+ + bx^- + \mu T_0 v^*(x) + \lambda T_1 v^*(x) \quad (10)$$

where  $T_0 v(x) = \min\{v(x), v(x + 1)\}$  and  $T_1 v(x) = \min\{v(x - 1), v(x) + c\}$ . As stated in the following proposition, the optimal policy can still be specified in terms of a production base-stock level and an order admission level. However, these levels are now fixed and no longer state-dependent.

**PROPOSITION 1:** In a system with a single customer class, an optimal policy can be specified in terms of a base-stock level  $s^*$  and an admission level  $w^*$ , with  $w^* \leq 0 \leq s^*$ , such that it is optimal to produce if  $x < s^*$  and not to produce otherwise and it is optimal to reject an order if  $x < w^*$  and, otherwise, to either satisfy it from on-hand inventory, if  $x > 0$ , or to backorder it, if  $x \leq 0$ .

Proposition 1 also applies to the average cost criterion. In that case, the optimal thresholds  $s^*$  and  $w^*$  can be obtained without solving the corresponding dynamic program. Recognizing that under a control policy with parameters  $(s, w)$ , the system behaves like a M/M/1/ $k$  make-to-stock queue, where  $k = |w| + s$ , the average cost rate is given by

$$J(s, w) = \lambda c \Pr(N = k) + hE[\max(s - N, 0)] + bE[\max(N - s, 0)] \quad (11)$$

where  $N$  is a random variable denoting the number of customers in a M/M/1/ $k$  queue. Using the fact that

$$\Pr(N = n) = \frac{(1 - \rho)\rho^n}{1 - \rho^{k+1}} \text{ for } n = 1, \dots, k,$$

where  $\rho = \lambda/\mu$ , the expression in (11) can be rewritten as

$$J(s, w) = \lambda c \frac{(1 - \rho)\rho^k}{1 - \rho^{k+1}} + h \sum_{n=0}^s (s - n) \frac{(1 - \rho)\rho^n}{1 - \rho^{k+1}} + b \sum_{n=s+1}^k (n - s) \frac{(1 - \rho)\rho^n}{1 - \rho^{k+1}}.$$

Recognizing that for fixed  $k$ , the distribution of  $N$  is independent of  $s$  and  $w$ , we can first choose optimal values for  $s$  and  $w$ , which we denote by  $s^*(k)$  and  $w^*(k)$ , and then choose the optimal value for  $k$ . For fixed  $k$ , the function  $J$  is convex in  $s$ . Therefore, the optimal base-stock level  $s^*(k)$  is given by the smallest integer for which  $J(s + 1, w|k) - J(s, w|k) \geq 0$ . This is equivalent to choosing the smallest  $s$  for which  $\Pr(N \leq s) \geq b/(b + h)$ , which leads to

$$s^*(k) = \left\lfloor \log \left( \frac{h + b\rho^{k+1}}{h + b} \right) / \log(\rho) \right\rfloor. \quad (12)$$

where the notation  $\lfloor x \rfloor$  denotes the integer floor of  $x$ . For fixed  $k$ , the optimal value of  $w$  is simply given by  $w^*(k) = -(k - s^*(k))$ . The optimal value of  $k$  can be obtained by substituting  $s$  in Eq. (11) with the value in Eq. (12) and then conducting an exhaustive search over a sufficiently large range of values for  $k$ .

##### 4.2. Make-to-Order Systems

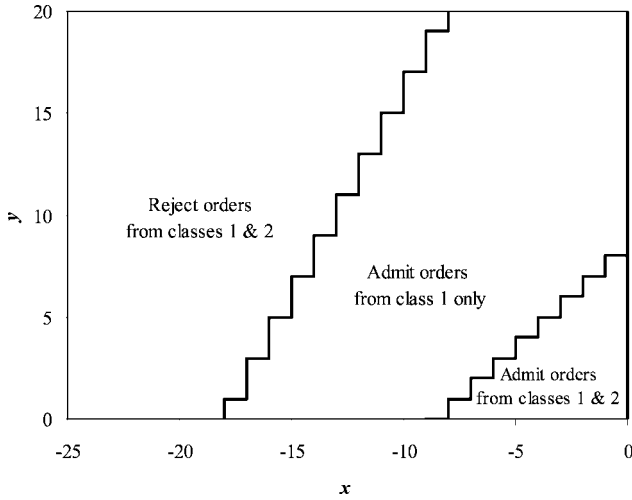
A make-to-order system is a system where the product cannot be held in inventory ahead of demand (e.g., when holding inventory is prohibitively expensive). In this case, there are only two types of decisions to be made, one regarding whether an incoming order is backordered or rejected and the other regarding the sequence in which orders are fulfilled. The optimality equation for this system is given by

$$v^*(x, y) = b_1 x^- + b_2 y + \mu T_0 v^*(x, y) + \lambda_1 T_1 v^*(x, y) + \lambda_2 T_2 v^*(x, y), \quad (13)$$

where the operators  $T_i, i = 0, 1, 2$ , are defined as follows:

$$T_0 v(x, y) = \begin{cases} \min\{v(x, y), v(x + 1, y), \\ v(x, y - 1)\} & \text{if } x < 0 \text{ and } y > 0 \\ \min\{v(x, y), v(x, y - 1)\} & \text{if } x = 0 \text{ and } y > 0 \\ \min\{v(x, y), v(x + 1, y)\} & \text{if } x < 0 \text{ and } y = 0 \\ v(x, y) & \text{if } x = 0 \text{ and } y = 0, \end{cases} \quad (14)$$





**Figure 6.** The structure of the optimal admission control policy ( $\mu = 1.00, \lambda_1 = 0.45, \lambda_2 = 0.45, c_1 = 500, c_2 = 250, b_1 = 10, b_2 = 5$ ).

$$T_1 v(x, y) = \min\{v(x - 1, y), v(x, y) + c_1\}, \quad (15)$$

and

$$T_2 v(x, y) = \min\{v(x, y + 1), v(x, y) + c_2\}. \quad (16)$$

Noting that it is never optimal not to produce whenever there is a backorder from either Class 1 or Class 2, operator  $T_0$  can be further simplified as

$$T_0 v(x, y) = \begin{cases} \min\{v(x + 1, y), & \text{if } x < 0 \text{ and } y > 0 \\ v(x, y - 1)\} & \text{if } x < 0 \text{ and } y > 0 \\ v(x, y - 1) & \text{if } x = 0 \text{ and } y > 0 \\ v(x + 1, y) & \text{if } x < 0 \text{ and } y = 0 \\ v(x, y) & \text{if } x = 0 \text{ and } y = 0, \end{cases}$$

As described in the following proposition (we omit the proof for the sake of brevity), and illustrated in Fig. 6, the optimal policy can be described in terms of two state-dependent admission levels  $w_1(y)$  and  $w_2(y)$ .

**PROPOSITION 2:** In a make-to-order system, an optimal policy can be specified in terms of two state-dependent admission levels  $w_1^*(y)$ , and  $w_2^*(y)$  such that it is optimal to admit orders from class  $i$  if  $w_i^*(y) < x$ , for  $i = 1, 2$ , and to reject the orders otherwise. Moreover,  $w_i^*(y)$  for  $i = 1, 2$  is nondecreasing in  $y$ .

Make-to-order systems have been studied in the literature in the context of optimal queue admission control. Although that literature is extensive, most of it deals with single class systems or systems where the backordering (delay) costs are the same for all the customer classes. In the case where the

backorder costs are identical,  $b_1 = b_2$ , the problem simplifies significantly (the state space can be described by the variable  $x$  only) and the optimal policy is specified by fixed admission levels (see Stidham [20] for an early reference and Çil et al. [29] for related discussion).

### 4.3. Other Special Cases

There are other problems with two customer classes that have been studied previously in the literature and that can be viewed as special cases of our problem. For example, in settings where backorders are not allowed, with orders either fulfilled from on-hand inventory or rejected, the optimal production policy can be shown (by suitably modifying the operators  $T_0, T_1$ , and  $T_2$ ) to be characterized by a fixed base-stock level that determines whether or not production takes place and a fixed inventory rationing level that determines whether orders from Class 2 are fulfilled from on-hand inventory or rejected; see Ha [16] for details. Similarly, for systems where lost sales are not allowed and all orders must be either fulfilled from on-hand inventory or backordered, the optimal policy consists of a fixed production base-stock level and a fixed inventory rationing level; see Ha [17] for a discussion of systems with two classes and de Véricourt et al. [18] for generalization to systems with multiple classes. We will revisit these two models in Section 5 on heuristics.

It is interesting to note that in the above cases, of either pure lost sales or pure backorders, the thresholds associated with production and inventory allocation are fixed and not state-dependent. In the case of pure lost sales, the intuition is simple: the state of the system can be described solely in terms of inventory level. Consequently, the optimality of a threshold policy immediately implies a fixed threshold. For the case of pure backorders, the system state is multidimensional in the same way it is in the more general case treated in the article. The optimality of state-independent thresholds is indeed surprising in that case. However, as noted by de Véricourt et al. ([18], p. 1493), this appears to be due to the linearity in the structure of the instantaneous cost in each state. This linearity does not hold when lost sales are allowed since depending on whether or not an order is rejected, there may or may not be a lost sales cost. As described by operators  $T_1$  and  $T_2$ , the decision of whether or not an order is rejected depends on a complex three-way comparison between fulfilling the order from inventory, backordering, or rejecting and incurring the lost sales cost.

Other variations that can be studied using our framework include hybrid make-to-order/make-to-stock systems where on-hand inventory is always reserved for one class (the make-to-stock class) while the other class is always either backordered or rejected (the make-to-order class). Carr and Duenyas [30] study the case where orders from the make-to-stock class must be rejected if no on-hand inventory is

available. For this case, the optimality equation simplifies as follows:

$$v^*(x, y) = hx + b_2y + \mu T_0 v^*(x, y) + \lambda_1 T_1 v^*(x, y) + \lambda_2 T_2 v^*(x, y), \quad (17)$$

where the operators  $T_i$  are now defined as follows:

$$T_0 v(x, y) = \begin{cases} \min\{v(x, y), v(x+1, y) \\ v(x, y-1)\} & \text{if } y > 0 \\ \min\{v(x, y), v(x+1, y)\} & \text{if } y = 0, \end{cases} \quad (18)$$

$$T_1 v(x, y) = \begin{cases} v(x-1, y) & \text{if } x > 0 \\ v(x, y) + c_1, & \text{otherwise,} \end{cases} \quad (19)$$

and

$$T_2 v(x, y) = \min\{v(x, y+1), v(x, y) + c_2\}. \quad (20)$$

Note that  $x$  now denotes on-hand inventory since no backorders from Class 1 are allowed. Also note that there are no decisions associated with operator  $T_1$  since orders from Class 1 cannot be backordered. Similarly, decisions associated with operator  $T_2$  are limited to either backordering or rejecting orders from Class 2 since orders from Class 2 cannot be fulfilled from on-hand inventory. Consequently, the optimal policy is specified by two state-dependent thresholds  $s(y)$  and  $w(y)$  such that it is optimal to produce to increase on-hand inventory if  $x < s(y)$ , where  $s(y)$  is decreasing in  $y$ , fulfill backorders from Class 2 if  $x < s(y)$  and  $y > 0$ , and admit orders from Class 2 if  $x > w(y)$ , where  $w(y)$  is increasing in  $y$ . This problem is obviously simpler than the one with which we deal in this article since there is no inventory allocation between the two classes and there is no problem of production sequencing, which arises when both classes are backordered.

## 5. HEURISTICS

In this section, we compare the performance of several plausible heuristics against the performance of the optimal policy for the general problem described in Sections 2 and 3. Our aim is to assess the benefit of using the optimal policy instead of simpler heuristics. We focus on heuristics that involve fixed (nonstate dependent) parameters since they are simpler to communicate and implement and, perhaps, are more common in practice. We provide a description of each of the heuristics we consider as follows:

### 5.1. Heuristic H1

Under heuristic H1, no rejection is allowed and all orders are either fulfilled from on-hand inventory or backordered.

The heuristic specifies two fixed thresholds, a base-stock level  $s_{H1}$  and an inventory rationing level  $r_{H1}$ . The production facility produces if  $x < s_{H1}$  and otherwise it does not. Orders from Class 1 are always fulfilled from on-hand inventory, if any is available, and otherwise are backordered. Orders from Class 2 are fulfilled from on-hand inventory if  $x \geq r_{H1}$  and otherwise are backordered. When there are backorders from both classes, priority is given to orders from Class 1. Backorders from Class 2 are fulfilled if  $x \geq r_{H1}$ . As we discussed in Section 4.3, this policy is optimal when rejection is not allowed. Optimal values for the parameters  $s_{H1}$  and  $r_{H1}$  can be obtained in closed form using the approach described in de Véricourt et al. [18].

### 5.2. Heuristic H2

Under Heuristic H2, no backorders are allowed. Orders from both classes are either fulfilled from on-hand inventory or rejected. Similar to H1, heuristic H2 specifies two fixed thresholds, a base-stock level  $s_{H2}$  and an inventory rationing level  $r_{H2}$ . The production facility produces if  $x < s_{H2}$  and otherwise it does not. Orders from Class 1 are always fulfilled from on-hand inventory, if any is available, and otherwise are lost. Orders from Class 2 are fulfilled from on-hand inventory if  $x \geq r_{H2}$  and otherwise are rejected. Ha [17] describes a search procedure for determining the optimal values for the parameters  $s_{H2}$  and  $r_{H2}$  using closed form expressions for the average cost.

### 5.3. Heuristic H3

Heuristic H3 allows for both backordering and rejection. The heuristic specifies two thresholds  $s_{H3}$  and  $w_{H3}$ . The production facility produces if  $x < s_{H3}$  and does not produce otherwise. Orders from both classes are either fulfilled from on-hand inventory, if any is available, on a first-come, first-served basis or backordered if  $x > w_{H3}$ . Orders are rejected regardless of class if  $x \leq w_{H3}$ . In contrast to H1 and H2, H3 does not differentiate between the two classes and orders from both classes are fulfilled on a first-come, first serve basis. This system can be analyzed using the single product model of Section 4.1 by letting  $b = \sum_{i=1}^2 p_i b_i$  and  $c = \sum_{i=1}^2 p_i c_i$  with  $p_i = \lambda_i / (\lambda_1 + \lambda_2)$ . The value of the parameters  $s_{H3}$  and  $w_{H3}$  can be obtained using the approach described in that section.

### 5.4. Heuristic H4

Similar to H3, heuristic H4 allows for both backordering and rejection. However, the heuristic differentiates between the two classes. To do so, H4 specifies four fixed thresholds  $s_{H4}$ ,  $r_{H4}$ ,  $w_{H4,1}$ , and  $w_{H4,2}$ . The production facility produces if  $x < s_{H4}$  and otherwise does not. Orders from Class 1

are either fulfilled from on-hand inventory or backordered if  $x > w_{H4,1}$ . Otherwise, they are rejected. Orders from Class 2 are fulfilled from on-hand inventory if  $x \geq r_{H4}$ . They are backordered if  $y \leq w_{H4,2}$  and are otherwise rejected. If both classes are backordered, the production facility produces first to fulfill backorders from Class 1, then to increase on-hand inventory up to  $r_{H4}$ , then to fulfill backorders from Class 2 when  $x \geq r_{H4}$ . Heuristic H4 mimics the optimal policy. The thresholds  $s_{H4}$ ,  $r_{H4}$ ,  $w_{H4,1}$ , and  $w_{H4,2}$  play similar roles to  $s^*(0)$ ,  $s^*(y)$ ,  $w_1^*(y)$ , and  $w_2^*(y)$  under the optimal policy. An important difference is of course the fact that the thresholds in H4 are not state-dependent but are instead fixed.

The average cost under the above heuristic for a given choice of the parameters  $s_{H4}$ ,  $r_{H4}$ ,  $w_{H4,1}$ , and  $w_{H4,2}$  can be computed using the following dynamic programming equation

$$v(x, y) + J_{H4}^* = hx^+ + b_1x^- + b_2y + \mu T_0^{H4}v(x, y) + \lambda_1 T_1^{H4}v(x, y) + \lambda_2 T_2^{H4}v(x, y),$$

where

$$T_0^{H4}v(x, y) = \begin{cases} v(x + 1, y) & \text{if } x < r_{H4} \\ v(x + 1, y) & \text{if } x < s_{H4} \text{ and } y = 0 \\ v(x, y - 1) & \text{if } x \geq r_{H4} \text{ and } y > 0, \end{cases}$$

$$T_1^{H4}v(x, y) = \begin{cases} v(x - 1, y) & \text{if } x > w_{H4,1}, \\ v(x, y) + c_1 & \text{otherwise,} \end{cases}$$

and

$$T_2^{H4}v(x, y) = \begin{cases} v(x - 1, y) & \text{if } x \geq r_{H4} \text{ and } y < w_{H4,2} \\ v(x, y + 1) & \text{if } x < r_{H4} \text{ and } y < w_{H4,2} \\ v(x, y) + c_2 & \text{if } y \geq w_{H4,2}. \end{cases}$$

The optimal parameters can be obtained using an exhaustive grid search. This search which can be extensive is facilitated by taking advantage of the fact that  $w_{H4,1} \leq 0 \leq r_{H4} \leq s_{H4}$ . Note that instead of the aforementioned dynamic programming formulation, the performance of the heuristic, for given values of the threshold parameters, could be evaluated using simulation.

### 5.5. Heuristic H5

Heuristic H5 is a less computing-intensive version of H4 (heuristic H4 is computing-intensive as there is no direct method to compute the thresholds). The heuristic specifies four fixed thresholds  $s_{H5}$ ,  $r_{H5}$ ,  $w_{H5,1}$ , and  $w_{H5,2}$  that play a similar role as the thresholds in heuristic H4. However, in this case, thresholds  $w_{H5,1}$  and  $w_{H5,2}$  are obtained by letting

$$w_{H5,i} = \frac{c_i/b_i}{\sum_{i=1}^2 c_i/b_i} w_{H5}$$

where  $w_{H5} = w_{H3}$  is obtained using the procedure described for heuristic H3 (H5 improves over H3 by associating customer-specific rejection thresholds that are increasing in  $c_i$  and decreasing in  $b_i$  for each customer class  $i$ ). Threshold  $r_{H5}$  is obtained using the direct procedure described for heuristic H1 so that  $r_{H5} = r_{H1}$ . The threshold  $s_{H5}$  is obtained by comparing the performance of the base-stock levels obtained for heuristics H1–H3, so that  $s_{H5} \in \{s_{H1}, s_{H2}, s_{H3}\}$ .

### 5.6. Heuristic H\*

Heuristic H\* is a composite heuristic that evaluates the performance of heuristics H1, H2, H3, and H5 and employs the one with the lowest cost. Note that the parameters of heuristics H1, H2, H3, and H5 are relatively easy to obtain, so that the overall computational effort remains modest.

To test the performance of the heuristics against that of the optimal policy, we carried out a series of numerical experiments for systems with a very wide range of parameter values. For the optimal policy (also for heuristic H4), numerical results are obtained by solving the corresponding dynamic programming equation using the value iteration method. The value iteration algorithm we use is a direct adaptation of the algorithm described in Puterman (Chapter 8, [31]). The state space is truncated at  $\{-x^{\min}, x^{\max}\} \times \{0, y^{\max}\}$  where  $x^{\min}$ ,  $x^{\max}$ , and  $y^{\max}$  are positive integers that are gradually increased until the cost is no longer sensitive to the truncation level. The value iteration algorithm is terminated once five-digit accuracy is obtained. For all the heuristics, we use the procedures described earlier to determine the corresponding optimal combination of control parameters. For heuristic H4, we limited the search for these parameters using the recurrent region for the optimal policy. This appears to be sufficient to support the conclusions we draw regarding the performance of H4. However, the performance of H4 may potentially improve with a more exhaustive search. For each scenario we tested, we obtain the average cost under the optimal policy  $J^*$  and the average cost under each of the heuristics  $J_{H_i}^*$  for  $i = 1, \dots, 5$  (also for H\*) and then compute the percentage difference between the cost of the optimal policy and the cost of each heuristic,  $(J_{H_i}^* - J^*)/J^*$ . We use the average cost criterion, instead of discounted cost, in our comparisons because the results are independent of the starting state and of the discount factor.

Representative results are shown in Tables 1–4. Based on these results, the following observations can be made.

- Heuristic H4 outperforms all other heuristics. The percentage cost difference between H4 and the optimal policy is generally small, with less than 2% in most of the cases tested. Heuristic H5 and H\* also perform well and in all cases are within 10% of the performance of the optimal policy. In contrast, heuristics

**Table 1.** The effect of backorder cost on the percentage cost difference between heuristics and the optimal policy.

$b_1/b_2$	H1	H2	H3	H4	H5	H*
1.0	35.82	35.07	9.97	0.36	1.55	1.55
1.1	35.61	34.43	10.42	0.38	1.50	1.50
1.2	35.41	33.79	11.22	0.41	1.40	1.40
1.4	35.01	32.54	12.16	0.39	1.32	1.32
1.6	34.61	31.30	12.86	0.36	1.44	1.44
1.8	34.23	30.09	13.11	0.31	1.95	1.95
2.0	33.99	29.04	13.92	0.29	2.47	2.47
3.0	33.03	24.64	14.68	0.53	3.75	3.75
4.0	32.33	22.48	16.08	1.36	6.00	6.00
5.0	31.72	20.45	16.26	1.39	5.78	5.78
10	30.60	14.77	16.33	0.99	5.67	5.67

$\mu = 1, \lambda_1 = 0.4, \lambda_2 = 0.5, c_1 = 500, c_2 = 250, h = 1, b_2 = 5.$

H1–H3 can perform poorly for certain combinations of parameter values.

- For H1, performance deteriorates when the backorder cost for one (or both) classes is high; see Table 1. This is consistent with intuition. Heuristic H1 does not have the option of limiting the number of backorders when backorder costs are high without increasing inventory. Consequently, cost can grow arbitrarily large with increases in backorder costs.
- For H2, performance deteriorates when the lost sale cost for one (or both) classes is high; see Table 2. Similar to H1, heuristic H2 does not have the option of limiting the number of lost sales without increasing inventory. Therefore, cost can also grow arbitrarily large with increases in the lost sales costs.
- For H3, performance deteriorates when the ratios  $b_1/b_2$  or  $c_1/c_2$  are high; see Tables 1 and 2. When these ratios are high, it becomes important to differentiate between the two classes by reserving inventory for Class 1 and giving Class 1 higher priority in fulfilling backorders. Heuristic H3 does not have this

**Table 2.** The effect of lost sales costs on the percentage cost difference between heuristics and the optimal policy.

$c_1/c_2$	H1	H2	H3	H4	H5	H*
1	36.08	18.75	3.86	0.82	1.63	1.63
1.1	34.46	19.28	4.03	0.26	0.74	0.74
1.2	34.11	20.89	4.93	0.25	0.89	0.89
1.4	33.99	24.34	7.19	0.28	1.65	1.65
1.6	33.99	25.98	9.53	0.29	2.48	2.48
1.8	33.99	27.51	11.88	0.29	3.32	3.32
2	33.99	29.04	13.92	0.29	2.47	2.47
3	33.99	33.84	21.44	0.29	3.14	3.14
4	33.99	36.88	27.83	0.29	3.01	3.01
5	33.99	39.62	32.11	0.29	4.97	4.97
10	33.99	45.70	44.39	0.29	7.63	7.63

$\mu = 1, \lambda_1 = 0.4, \lambda_2 = 0.5, c_2 = 250, h = 1, b_1 = 10, b_2 = 5.$

**Table 3.** The effect of holding cost on the percentage cost difference between heuristics and the optimal policy.

$h$	H1	H2	H3	H4	H5	H*
0.1	24.10	11.19	8.23	0.18	2.22	2.22
0.2	27.22	14.07	9.40	0.29	2.59	2.59
0.3	29.11	16.43	10.52	0.21	2.69	2.69
0.4	30.44	18.62	11.04	0.24	3.11	3.11
0.5	31.41	20.50	11.65	0.30	1.99	1.99
1	33.99	29.04	13.92	0.29	2.47	2.47
1.5	35.20	36.45	15.33	0.35	2.84	2.84
2	35.77	43.39	16.53	0.32	2.29	2.29
3	36.37	55.92	18.17	0.32	4.89	4.89
5	36.83	78.39	20.19	0.41	3.37	3.37
10	36.82	124.29	26.08	0.46	6.95	6.95

$\mu = 1, \lambda_1 = 0.4, \lambda_2 = 0.5, c_1 = 500, c_2 = 250, b_1 = 10, b_2 = 5.$

ability and the system is forced to either hold more inventory or incur the higher backordering and lost sales costs associated with Class 1.

- For heuristics H2 and H3, relative performance deteriorates when the unit holding cost increases; see Table 3. For H1, H4, H5 (and also H\*), relative performance remains somewhat unaffected. These heuristics can mitigate the effect of higher holding cost by having the system hold less inventory and backordering more (this is similar to what the optimal policy does). This is not possible under H2 where lower inventory leads to higher lost sales. Backordering is possible under H3. However, H3 cannot ration inventory among the two classes nor can it give higher priority to Class 1 when fulfilling backorders, the way H4, H5, and H\* can.
- The percentage difference in cost between the optimal policy and all heuristics other than H1 is relatively small when the demand workload, as measured by the ratio  $\rho = (\lambda_1 + \lambda_2)/\mu$ , is high; see Table 4. In contrast, the percentage cost difference is large for H1.

**Table 4.** The effect of workload on the percentage cost difference between heuristics and the optimal policy.

$\rho$	H1	H2	H3	H4	H5	H*
0.6	0.00	110.13	8.53	0.00	0.30	0.00
0.7	0.13	86.83	16.08	0.01	1.28	0.13
0.8	3.38	55.21	11.10	0.13	1.80	1.80
0.9	33.99	29.04	13.92	0.29	2.47	2.47
0.92	53.22	25.34	14.38	0.36	4.29	4.29
0.94	86.34	21.97	15.37	0.42	3.98	3.98
0.96	154.77	18.97	16.12	0.26	3.98	3.98
0.98	364.31	16.19	17.27	0.33	7.94	7.94
0.99	784.80	14.86	17.92	0.22	8.18	8.18

$\mu = 1, \lambda_1 = 0.4, \lambda_2 = 0.5, \rho = \alpha(\lambda_1 + \lambda_2)/\mu, \rho$  is varied by varying  $\alpha, c_1 = 500, c_2 = 250, h = 1, b_1 = 10, b_2 = 5.$

For heuristics other than H1, as well as for the optimal policy, total cost is dominated by the lost sales costs when workload is high, which perhaps explains the small percentage cost difference between these heuristics and the optimal policy. For H1, rejecting customers is not possible, so that the system must incur high backorder costs when the utilization of the production facility is high. In fact, under H1, the number of backorders grows exponentially as  $\rho$  approaches 1 and so does the cost. The costs under the other policies are however always bounded by  $c_1\lambda_1 + c_2\lambda_2$ , which is the cost of a policy that never holds inventory and rejects all demand.

In summary, the numerical results suggest that heuristic H4 can be an effective substitute to the optimal policy for a wide range of parameter values. This is not surprising given that H4 has several of the same features as the optimal policy, including the ability to limit both backorders and rejections and to differentiate between the two customer classes. Heuristics H5 and H\* are also possible substitutes for the optimal policy, particularly when computational effort is a concern. Other heuristics that allow for either only backordering or only lost sales, or do not differentiate between customer classes, can perform poorly. For managers in practice, these results highlight the importance of permitting (but also limiting) both backorders and lost sales and differentiating between customer classes. The corresponding cost savings can be significant.

## 6. SUMMARY AND CONCLUDING COMMENTS

In this article, we studied a production inventory-system with a single product and two customer classes where both backorders and lost sales are permitted. We formulated the problem as a Markov decision process and used it to characterize the structure of the optimal policy. We showed that the optimal policy limits the amount of backorders from each class and does so dynamically by taking into account the current state of the system. We also showed that the optimal policy differentiates between the classes in deciding whether or not to fulfill orders from a particular class and, if an order is to be fulfilled, whether or not it should be fulfilled immediately from on-hand inventory or backordered. These decisions are made based on thresholds that also depend on the current state of the system. Using numerical results, we compared the performance of the optimal policy against several heuristics and showed that those that do not allow for the possibility of both backordering and rejecting or do not differentiate between the customer classes can perform poorly. We found that heuristics that mimic the optimal policy but use fixed inventory and

admission thresholds can be effective substitutes to the more complex optimal policy for a wide range of parameters.

There are several potential avenues for future research. It will be useful to consider systems with different demand and production time distributions. For example, it is possible to substitute the exponential distribution by phase-type distributions which can be constructed to approximate other more general distributions. The use of phase-type distributions retains the Markovian property of the system and continues to allow the formulation of the problem as an MDP. The drawback is that the dimensions of the problem increase with the number of phases. In turn, this could make the analysis less tractable and the optimal policy more difficult to characterize. It will also be useful to extend the analysis to systems with an arbitrary number of customer classes. Here too, we expect the analysis to become significantly less tractable because of the multidimensionality of the problem. A state variable would need to be associated with the backorder from each class. Nevertheless, we expect the structure of the optimal policy to remain the same with inventory rationing and admission thresholds associated with each customer class. These thresholds would now be dependent on the vector of backorder levels from all the classes.

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