On the Impact of Input Price Variability and Correlation in Stochastic Inventory Systems

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Abstract

In this paper, we explore the impact of input price variability and correlation in the context of an inventory system with stochastic demand and stochastic input prices. For a general class of such systems, we show that the expected cost function is concave in the input price. From this, it follows that higher input price variability in the sense of the convex order always leads to lower expected cost. We show that this is true under a wide range of assumptions for price evolution, including cases with i.i.d. prices and cases where prices are correlated and evolve according to an AR(1) process, a geometric Brownian motion, or a Markovian martingale. In addition, the result holds in cases where there is just a single period. We also examine the impact of price correlation over time and across inputs, and we find that expected cost is increasing in price correlation over time and decreasing in price correlation across components. We present results of a numerical study that provide insights on how various parameters influence the effects of price variability and correlation.

Keywords: inventory systems; stochastic prices; dynamic programming; price variability; price correlation; stochastic orders

1 Introduction

Stochastic input prices are common in practice. The prices of raw materials, precious metals, grain commodities, and electronic components, among many others, can fluctuate considerably over short periods. Such fluctuations may result from variations in supply and demand, changes in market conditions, or the introduction of new technology. Firms in some industries face input price variability because of their reliance on spot markets for procurement and, in the case of firms with global supply chains, because of exchange rate fluctuation.

The presence of stochastic input prices raises several important questions. First, how does the presence of variability in input prices affect input ordering decisions and the nature of the optimal ordering policy? Second, how does price variability affect performance, and particularly cost? Does higher price variability increase or decrease overall costs? How does price correlation, over time or across inputs, interact with price variability and what is the net effect on cost? Is the effect of price variability more pronounced with higher correlation?

There is literature dealing with inventory systems with stochastic input prices; see Zhang (2012) for a comprehensive review. In a periodic review inventory system, Kalymon (1971) considers a single-item model with setup costs in which future input prices are determined by a Markovian stochastic process, and establishes that the optimal policy is a price-dependent (s, S) policy. Golabi (1985) considers a problem with an independent price process, negligible setup cost, and deterministic demand. He shows that the optimal policy is to always purchase a quantity that covers demands for the next several periods, and that this number of periods is decreasing in the current price. Gavirneni (2004) develops an efficient recursive procedure to calculate the base stock level when there are no setup costs and shows that myopic solutions are very effective under a non-speculative assumption. For continuous review inventory systems, Song and Zipkin (1993) characterize the optimal policy and develop algorithms for settings with Markov modulated purchasing price and Markov modulated demand. Yang and Xia (2009) consider a problem in which the input price follows a discrete-state Markov process and demand is a compound Poisson process. They show that the optimal policy is of the order-up-to type and identify conditions under which the order-up-to levels are decreasing in price. Berling and Martínez-de-Albéniz (2011) study a problem in which the price evolution is a continuous stochastic process and demand is Poisson.

They characterize the optimal base-stock level using a series of threshold prices. Nie et al. (2014) consider a firm buying raw material from the spot market and selling a final product by submitting bids. They show that the optimal procurement policy is a price-dependent base-stock policy and the optimal bidding price decreases in the inventory level.

In the finance literature, Gibson and Schwartz (1990), Schwartz and Smith (2000), and Casassus and Collin-Dufresne (2005) develop multi-factor models to describe the dynamics of commodity prices. They test these models using empirical data and discuss implications for option valuation and investment decisions. There is a growing body of operations management literature concerned with traded commodities. This literature characterizes optimal operating policies for traded commodities regarding how much to buy, produce, and sell; see for example, Martínez-de-Albéniz and Simón (2009), Secomandi (2010), Devalkar et al. (2011), Goel and Gutierrez (2006, 2011), and Guo et al. (2011)

Another stream of literature studies the impact of a spot market on supply chain operations. Yi and Scheller-Wolf (2003), Boyabatlı et al. (2011), Inderfurth and Kelle (2011), Chen et al. (2013), and Secomandi and Kekre (2014) consider models in which a firm can procure a resource through long-term contracts or from the spot market. They characterize the optimal procurement policy under different assumptions. Park et al. (2012) study the inventory sharing problem for two firms where the firms can procure the commodity and sell excess inventory through either the spot or forward market. They show that inventory sharing is always beneficial. Haksöz and Seshadri (2007) provide a comprehensive review on the use of spot market operations to manage procurement in supply chains.

Much of this literature is concerned with describing the structure of the optimal ordering policy or with identifying other effective heuristics. There is only limited literature that studies the impact of input price variability in the context of inventory systems. Ho et al. (1998) analyze the impact of price format (the average price and the variance of the price) on the shopping frequency and purchasing behavior of a rational shopper using an economic order quantity (EOQ) model. They show that the optimal long run average cost is decreasing with the price variance. Berling and Rosling (2005) study how financial risks influence the optimal value of the order quantity and the reorder level in an inventory system with setup costs. They show that the systematic risk of demand has a negligible effect, but the systematic risk of the purchase price has a significant effect. Plambeck and Taylor (2013) study a problem where the firm is a price taker for both input and output products. They show that input price variability reduces the value of improving input efficiency (output produced per unit input) but increases that of capacity efficiency (the rate at which a production facility can convert input into output). Output price variability increases the value of capacity efficiency, but it increases the value of input efficiency only under certain conditions.

The papers by Janakiraman and Seshadri (2011) and Boyabatlı et al. (2011) are the most relevant to our study. Janakiraman and Seshadri (2011) examine a family of dynamic programs with stochastic cost parameters in which the vector of cost parameters evolves as a stochastic process. They show that if the single period cost is concave with respect to this vector, then the optimal cost is bounded above by the optimal cost for the dynamic program in which these stochastic cost parameters are replaced by their expectations in each period. However, the approach they employ cannot be used to compare two dynamic programs each with stochastic cost parameters. Boyabatlı et al. (2011) study optimal procurement, processing, and production policies for a meat-processing company which sources input through long-term contracts and from a spot market. They assume that the spot price follows a normal distribution and show that the optimal expected profit of the firm increases in the spot price variability under certain conditions.

In the economics literature, there is a stream of research that examines a firm's behavior when price or cost fluctuates. Sandmo (1971) and Batra and Ullah (1974) study the optimal output and input decisions for a competitive firm under price uncertainty and risk aversion. Anderson and Danthine (1981, 1983), Meyer (1987) and Kamara (1993) study how firms can use futures to hedge or speculate against price uncertainty. This literature relies on aggregate models of demand and supply and does not model operational decisions.

In this paper, we show that for a wide range of inventory problems and assumptions, higher input price variability (as measured by convex ordering of prices) leads to lower expected inventory costs over the planning horizon, where inventory costs include ordering, inventory holding, and shortage costs. One may initially attribute this phenomenon to the fact that higher variability affords more frequent opportunities to place large (small) orders in periods in which prices are anticipated to be higher (lower) in subsequent periods. Although we do observe such a period-over-period effect, the main result also holds when the input prices evolve as a martingale, where the price in a current period is equal to the conditional expected price in future periods. In addition, the result holds in systems with a single period where ordering decisions cannot be postponed to the future. We show that the benefit of input price variability can be traced to the concavity of the cost function with respect to the input price. This concavity in price is a consequence of the ability of the system manager to adjust the order quantity as prices change, leading to a cost that is lower than that which would be incurred if the order quantity were left unchanged.

We also examine the impact of correlation of prices over time. For certain types of input price sequences, we show that the expected cost decreases with increases in input price correlation. We also consider inventory systems with multiple inputs and allow for correlation among the prices of different inputs. For such systems, we use the notion of supermodular ordering to show that the expected total cost is decreasing in the correlation in input prices. Finally, we present numerical results illustrating how the benefit of input price variability is affected by various parameters. These results suggest, for instance, that the magnitude of the benefit of price variability is increasing in the length of the planning horizon and the correlation of prices of different inputs, and decreasing in the holding and backorder costs and the correlation in prices over time. The numerical results also suggest that the impact of price correlation over time and across components is more significant when the price variability is higher.

The rest of the paper is organized as follows. In section 2, we describe and formulate the single-component inventory model and describe the structure of the optimal policy. In section 3, we analyze the impact of input price variability. In section 4, we study the impact of correlation of the input prices over time. In section 5, we consider inventory systems with multiple inputs and for such systems we study the impact of input price variability and the impact of correlation across component prices. In section 6, we provide numerical results and explore some of the implications of the results. In section 7, we provide concluding comments.

2 Problem Formulation

We consider a multi-period stochastic inventory control problem for a single product over a finite planning horizon consisting of $T \geq 1$ discrete time periods. Time $t = 1$ is the first period and time $t = T$ is the last period. Demand for the product occurs each period. We assume that demand forms an i.i.d. sequence of random variables with common distribution function $\Phi(\cdot)$ and density function $\phi(\cdot)$. We assume that one unit of the product is needed to fulfill one unit of demand. In each period, the ordering price, to which we also refer as the *input price*, is stochastic as well and is realized at the beginning of the period, before the realization of demand. An ordering decision (whether or not and how much to order) is made at the beginning of each period before the realization of demand but after the realization of the input price. There is no leadtime (the extension to positive leadtime is straightforward), and therefore quantities ordered in a period, if any, can be used to fulfill demands in that same period. Each unit of positive leftover inventory at the end of a period incurs a holding cost of h. Unfulfilled demand is backlogged and a backorder cost of b per unit backlogged per period is incurred. The one-period discount factor is denoted by $\beta \in (0, 1]$.

We assume that the sequence of ordering prices $\{X_t : t = 1, \ldots, T\}$ follows a Markov chain, where the ordering price X_{t+1} in period $t + 1$ depends on the ordering price X_t in period t and another random variable ϵ_t. Specifically, we assume that

$$X_{t+1} = f_t(\epsilon_t)X_t + g_t(\epsilon_t), \qquad t = 1, \ldots, T - 1, \tag{1}$$

where $\{\epsilon_t : t = 1, \ldots, T - 1\}$ is a sequence of independent random variables. We denote the distribution function of ϵ_t as $\Psi_t(\cdot)$. We assume that the sequence $\{\epsilon_t\}$, the initial input price X_1, and the sequence of demands are mutually independent.

This assumption about price evolution is quite general. For instance, the case of i.i.d. ordering prices can be obtained by taking $f_t(\epsilon) = 0$, $g_t(\epsilon) = \epsilon$, and $\{\epsilon_t\}$ i.i.d. with the same distribution as X_1. Other special cases of (1) include a discrete-time analog of geometric Brownian motion as well as auto-regressive processes of order 1 (AR(1) processes). To obtain geometric Brownian motion with drift μ and volatility σ, we take $\{\epsilon_t\}$ to be i.i.d. normal random variables with mean μ and variance σ^2, $f_t(\epsilon) = e^\epsilon$, and $g_t(\epsilon) = 0$, in which case equation (1) becomes $X_{t+1} = X_t e^{\epsilon_t}$. To obtain an AR(1) process, we take $f_t(\epsilon) = \rho_t$, $g(\epsilon) = \epsilon + c_t$, and $E\epsilon_t = 0$, in which case (1) becomes $X_{t+1} = c_t + \rho_t X_t + \epsilon_t$. Moreover, through appropriate choices of $\{\epsilon_t\}$, $f_t(\cdot)$, and $g_t(\cdot)$, we can make the sequence of prices $\{X_t\}$ a martingale, supermartingale, or submartingale. We will discuss all these examples later. From here on, for notational simplicity, we only consider the case where $f_t(\cdot) = f(\cdot)$ and $g_t(\cdot) = g(\cdot)$. Our results also apply to cases where $f_t(\cdot)$ and $g_t(\cdot)$ or the holding and backorder cost parameters are time heterogeneous.

In view of the preceding assumptions, the problem can be viewed as a Markov decision process where the state of the system at the beginning of each period is a pair (s, x) that represents the net inventory level s and the ordering price x. In each period, the action, i.e., the decision to be made, is the order-up-to level $y \in [s, \infty)$. If in a particular period, net inventory is s, order-up-to level y is chosen, and demand is ξ, then the order quantity is $y - s$ and the net inventory level in the subsequent period is $y - \xi$.

The expected one-period holding and shortage costs can be expressed as a function of the action y as follows:

$$L(y) = \int_0^y h(y-\xi)\phi(\xi)d\xi + \int_y^\infty b(\xi - y)\phi(\xi)d\xi.$$

The objective is to determine in each period the optimal order-up-to level for each price such that the expected total discounted cost over the planning horizon is minimized. For $t = 1, \ldots, T$, let $v_t(s, x)$ be the optimal expected total cost from period t onward when the net inventory at the beginning of period t is s and the ordering price in period t is x. The optimality equations are given by

$$\begin{aligned} v_t(s,x) &= \min_{y \geq s} \left\{ x(y-s) + L(y) + \beta \int_\xi \int_\epsilon v_{t+1}(y-\xi, f(\epsilon)x + g(\epsilon))\Psi_t(d\epsilon)\phi(\xi)d\xi \right\} \\ &= \min_{y \geq s}\{w_t(y,x)\} - xs, \end{aligned} \tag{2}$$

where

$$\begin{aligned} w_t(y,x) &= xy + L(y) + \beta \int_\xi \int_\epsilon v_{t+1}(y-\xi, f(\epsilon)x + g(\epsilon))\Psi_t(d\epsilon)\phi(\xi)d\xi, \\ &= xy + L(y) + \beta \int_\xi E[v_{t+1}(y-\xi, X_{t+1})|X_t = x]\phi(\xi)d\xi \end{aligned} \tag{3}$$

and

$$v_{T+1}(s,x) = 0.$$

We let $y_t^*(s,x)$ denote a minimizer of (2). Then an optimal policy uses order-up-to level $y_t^*(s,x)$ if the state is (s,x) in period t, and the optimal order quantity is $y_t^*(s,x)-s$. The optimal expected total cost for the entire planning horizon (computed before learning the first ordering price) with starting inventory s is given by $V_1(s) = Ev_1(s, X_1)$.

In preparation for our analysis of the impact of input price variability, we next describe the form of the optimal policy for this inventory system. We begin with the following lemma.

Lemma 1. *The function $w_t(y,x)$ is convex in y for all x and $t = 1, \ldots, T$.*

The proof of Lemma 1 (and all other proofs not provided in the paper) can be found in the appendix. Let $y_t^\circ(x)$ denote a minimizer of $w_t(y,x)$ over $y \in (-\infty, \infty)$. An optimal policy is described in the following proposition, which follows immediately from Lemma 1.

Proposition 1. *There exists an optimal ordering policy for the multi-period inventory system with stochastic input prices that is a state-dependent base stock policy with base stock levels $y_t^\circ(x)$. That is, $y_t^*(s,x) = \max\{s, y_t^\circ(x)\}$ and the optimal order quantity in state (s,x) at time t is $\max\{0, y_t^\circ(x) - s\}$.*

The optimal base stock level $y_t^\circ(x)$ need not be decreasing in the realized price x. For example, consider a case where $T = 2$, $b = 0.5, h = 0.5$, $D_1 = D_2 = 10$ and $X_2 = 2X_1 - 5$, and suppose that the marginal distribution for the ordering price in period 1 is $P(X_1 = 4) = P(X_1 = 6) = 0.5$ and thus the marginal distribution of the ordering price in period 2 is $P(X_1 = 3) = P(X_1 = 7) = 0.5$. In this case, it is easy to check that it is optimal to order nothing if the realized ordering price in period 1 is 4 ($y_1^\circ(4) = 0$) and to order up to 20 if the realized ordering price in period 1 is 6 ($y_1^\circ(6) = 20$). Therefore, the optimal base stock is increasing with respect to the realized price. This is due to the strong positive correlation in the ordering price across periods. In the following proposition, we provide a sufficient condition under which this phenomenon does not occur and the base stock level is decreasing in the realized price.

Proposition 2. *If $E|f(\epsilon_t)| \leq 1$ for $t = 1, \ldots, T$, then $y_t^\circ(x)$ is decreasing in x for $t = 1, \ldots, T$.*

Examples that satisfy the condition $E|f(\epsilon_t)| \leq 1$ for $t = 1, \ldots, T$ include the case of i.i.d. input prices and the case where the input prices evolve according to an AR(1) process. In the first case $f(\epsilon) = 0$, and in the second case $f(\epsilon) = \rho \in [-1, 1]$. If the condition in the proposition is not satisfied, for example, if $Ef(\epsilon_t) > 1$, then it is possible that a high (low) price in one period would lead to a even higher (lower) expected price in the next period. In this case, it may be optimal to order more (less) when the price is high (low). Or, if $Ef(\epsilon_t) < -1$, then a low price in one period (say, now) would lead to a high expected price in the next period and an even lower expected price after two periods. In that case, one may wish to order more now in anticipation of a high price in the next period but to order less now in anticipation of an even lower price after two periods. It is possible that the second of these two effects is stronger. Therefore, it is possible that it is optimal to order more as price increases.

3 Impact of Price Variability

In this section, we discuss the impact of input price variability on the expected total cost and show that higher variability yields lower expected total cost. In our analysis, we use the tool of convex ordering to compare different levels of price variability. A random variable X is said to be smaller than $\widehat{X}$ in the *convex order* (written $X \leq_{cx} \widehat{X}$) if $Eu(X) \leq Eu(\widehat{X})$ for all convex functions $u(\cdot)$ such that the expectations exist. The concept of convex order is reviewed in, for example, Müller and Stoyan (2002) and Shaked and Shanthikumar (2007). If $X \leq_{cx} \widehat{X}$, then it is well known that $EX = E\widehat{X}$ and $\text{Var}(X) \leq \text{Var}(\widehat{X})$. For random variables drawn from various common distributions, convex ordering is equivalent to having ordered variances and identical means. For example, if we compare two normal random variables with the same mean, then the one with the smaller variance is smaller in the convex order. The same holds true for uniform, gamma and lognormal random variables as well. Below, we will frequently make use of the fact that if $u(\cdot)$ is concave and $X \leq_{cx} \widehat{X}$ then $Eu(X) \geq Eu(\widehat{X})$.

The next lemma establishes the concavity of the cost function $v_t(s, x)$ with respect to the ordering price x.

Lemma 2. *$v_t(s, x)$ is concave in x for all s and $t = 1, \ldots, T + 1$.*

To study the impact of ordering price variability, we compare two different inventory systems with ordering price sequences $\{X_t\}$ and $\{\widehat{X}_t\}$ and noise sequences $\{\epsilon_t\}$ and $\{\widehat{\epsilon}_t\}$ satisfying $X_{t+1} = f(\epsilon_t)X_t + g(\epsilon_t)$ and $\widehat{X}_{t+1} = f(\widehat{\epsilon}_t)\widehat{X}_t + g(\widehat{\epsilon}_t)$ respectively. All other parameters of the two systems are the same. We assume that each of the two systems individually satisfies the assumptions after (1) in Section 2. Let $\widehat{v}_t(s,x)$ be the optimal expected total cost-to-go in period t when the inventory is s and the realization of price is x for the system with ordering prices $\{\widehat{X}_t\}$.

In preparation for our next result, let $X_{t+1}(x) = f(\epsilon_t)x + g(\epsilon_t)$ be a random variable that follows the conditional distribution of X_{t+1} given $X_t = x$. Likewise, let $\widehat{X}_{t+1}(x) = f(\widehat{\epsilon}_t)x + g(\widehat{\epsilon}_t)$ be a random variable that follows the conditional distribution of $\widehat{X}_{t+1}$ given $\widehat{X}_t = x$. With this notational device, $w_t(y,x)$ in (3) can be written as

$$w_t(y,x) = xy + L(y) + \beta \int_{\xi} E[v_{t+1}(y-\xi, X_{t+1}(x))]\phi(\xi)d\xi. \tag{4}$$

The following theorem describes the impact of price variability on the optimal expected total cost.

Theorem 1. *Consider $k \in \{1,\ldots,T-1\}$ and suppose $X_{t+1}(x) \leq_{cx} \widehat{X}_{t+1}(x)$ for all x and $t = k,\ldots,T-1$. Then $v_t(s,x) \geq \widehat{v}_t(s,x)$ for all (s,x) and $t = k,\ldots,T+1$ and $E[v_t(s,X_t)|X_{t-1} = x] \geq E[\widehat{v}_t(s,\widehat{X}_t)|\widehat{X}_{t-1} = x]$ for all x and $t = k+1,\ldots,T$.*

Proof. For a given $k = 1,\ldots,T-1$, we first prove that $v_t(s,x) \geq \widehat{v}_t(s,x)$ for all (s,x) and $t = k,\ldots,T+1$ by induction on t. We have $v_{T+1}(s,x) = 0 = \widehat{v}_{T+1}(s,x)$. Suppose $v_{t+1}(s,x) \geq \widehat{v}_{t+1}(s,x)$ for all (s,x). Then by (2) and (4), we have

$$\begin{aligned}
v_t(s,x) &= \min_{y \geq s} \left\{ x(y-s) + L(y) + \beta \int_{\xi} E[v_{t+1}(y-\xi, X_{t+1}(x))]\phi(\xi)d\xi \right\} \\
&\geq \min_{y \geq s} \left\{ x(y-s) + L(y) + \beta \int_{\xi} E[v_{t+1}(y-\xi, \widehat{X}_{t+1}(x))]\phi(\xi)d\xi \right\} \\
&\geq \min_{y \geq s} \left\{ x(y-s) + L(y) + \beta \int_{\xi} E[\widehat{v}_{t+1}(y-\xi, \widehat{X}_{t+1}(x))]\phi(\xi)d\xi \right\} \\
&= \widehat{v}_t(s,x).
\end{aligned}$$

The first inequality above follows from the assumption that $X_{t+1}(x) \leq_{cx} \widehat{X}_{t+1}(x)$ and the fact that $v_{t+1}(s,x)$ is a concave function of x as shown in Lemma 2. The second inequality above follows from the inductive hypothesis. Thus, $v_t(s,x) \geq \widehat{v}_t(s,x)$ for all s, x, and $t = k,\ldots,T+1$. For

$t = k+1, \ldots, T$ we have

$$E[v_t(s, X_t)|X_{t-1} = x] = Ev_t(s, X_t(x)) \geq Ev_t(s, \widehat{X}_t(x)) \geq E\widehat{v}_t(s, \widehat{X}_t(x)) = E[\widehat{v}_t(s, \widehat{X}_t)|\widehat{X}_{t-1} = x],$$

where the first inequality uses Lemma 2. □

The following corollary is an immediate consequence of Theorem 1, because for a given $k = 1, \ldots, T-1$, if $X_k \leq_{cx} \widehat{X}_k$, then $Ev_k(s, X_k) \geq E\widehat{v}_k(s, X_k) \geq E\widehat{v}_k(s, \widehat{X}_k)$, where the second inequality is due to the fact that $\widehat{v}_k(s, x)$ is concave in x.

Corollary 1. *Consider $k \in \{1, \ldots, T-1\}$ and suppose $X_{t+1}(x) \leq_{cx} \widehat{X}_{t+1}(x)$ for all x, and $t = k, \ldots, T-1$. If $X_k \leq_{cx} \widehat{X}_k$, then $Ev_k(s, X_k) \geq E\widehat{v}_k(s, \widehat{X}_k)$. In particular, if $X_1 \leq_{cx} \widehat{X}_1$, then $Ev_1(s, X_1) \geq E\widehat{v}_1(s, \widehat{X}_1)$.*

In view of the assumption of Markovian ordering prices, Theorem 1 indicates that given a history of the price realizations, the optimal expected total cost-to-go is decreasing with respect to the conditional variability of subsequent ordering prices. Corollary 1 shows that if no information is known about past prices, the unconditional optimal expected total cost-to-go is decreasing with respect to the unconditional variability of the current ordering price. In both cases, the more variable the price is, the lower the optimal expected total cost is. Our result implies that a risk neutral decision maker has a preference for suppliers with high price variability over suppliers with low price variability or suppliers with fixed prices. This contrasts with the effect of demand variability, where in many inventory systems, greater variability in demand leads to higher expected cost (or lower expected profit).

In the following proposition, we provide conditions under which the assumptions of Theorem 1 and Corollary 1 hold.

Proposition 3. *Suppose $X_1 \leq_{cx} \widehat{X}_1$. Then, the following statements hold.*

(a) If $\epsilon_t = \widehat{\epsilon}_t$ for $t = 1, \ldots, T$, then $X_t(x) \leq_{cx} \widehat{X}_t(x)$ for all x and $X_t \leq_{cx} \widehat{X}_t$ for $t = 2, \ldots, T$.

(b) If $\epsilon_t \leq_{cx} \widehat{\epsilon}_t$ and $f(\cdot)$ and $g(\cdot)$ are convex functions such that $Ef(\epsilon_t) = Ef(\widehat{\epsilon}_t)$ and $Eg(\epsilon_t) = Eg(\widehat{\epsilon}_t)$ for $t = 1, \ldots, T$, then $X_t(x) \leq_{cx} \widehat{X}_t(x)$ for $x \geq 0$ and $t = 2, \ldots, T$. Moreover, if X_t or $\widehat{X}_t$ is nonnegative a.s. for $t = 1, \ldots, T$, then $X_t \leq_{cx} \widehat{X}_t$ for $t = 2, \ldots, T$.

(c) *If* $\epsilon_t \leq_{cx} \widehat{\epsilon}_t$ *and* $g(\cdot)$ *is a convex function such that* $Eg(\epsilon_t) = Eg(\widehat{\epsilon}_t)$ *for* $t = 1, \ldots, T$*, and* $f(\cdot)$ *is an affine function, then* $X_t(x) \leq_{cx} \widehat{X}_t(x)$ *for all* x *and* $X_t \leq_{cx} \widehat{X}_t$ *for* $t = 2, \ldots, T$.

(d) *If* $f(\epsilon_t) \leq_{cx} f(\widehat{\epsilon}_t)$ *for* $t = 1, \ldots, T$ *and* $g(\cdot)$ *is a constant, then* $X_t(x) \leq_{cx} \widehat{X}_t(x)$ *for all* x *and* $X_t \leq_{cx} \widehat{X}_t$ *for* $t = 2, \ldots, T$.

Proof. (a) Suppose $u(\cdot)$ is an arbitrary convex function. Then

$$Eu(X_{t+1}(x)) = Eu(f(\epsilon_t)x + g(\epsilon_t)) = Eu(f(\widehat{\epsilon}_t)x + g(\widehat{\epsilon}_t)) = Eu(\widehat{X}_{t+1}(x)).$$

Therefore, $X_{t+1}(x) \leq_{cx} \widehat{X}_{t+1}(x)$ for all x and $t = 2, \ldots, T$.

To show that $X_t \leq_{cx} \widehat{X}_t$ for $t = 2, \ldots, T$, we only need to show that if $X_t \leq_{cx} \widehat{X}_t$, then $X_{t+1} \leq_{cx} \widehat{X}_{t+1}$. Suppose $X_t \leq_{cx} \widehat{X}_t$. Let $u(\cdot)$ be an arbitrary convex function and let $\kappa(x) = Eu(X_{t+1}(x))$ and $\widehat{\kappa}(x) = Eu(\widehat{X}_{t+1}(x))$. Then $\kappa(x) \leq \widehat{\kappa}(x)$ for all x because $u(\cdot)$ is convex and we have already shown that $X_{t+1}(x) \leq_{cx} \widehat{X}_{t+1}(x)$ for all x. Moreover, $\kappa(x) = Eu(f(\epsilon_t)x + g(\epsilon_t))$ and hence κ is convex in x. Therefore, we have

$$Eu(X_{t+1}) = E\kappa(X_t) \leq E\kappa(\widehat{X}_t) \leq E\widehat{\kappa}(\widehat{X}_t) = Eu(\widehat{X}_{t+1}).$$

Thus, we have $X_{t+1} \leq_{cx} \widehat{X}_{t+1}$.

(b) We will use the fact that $X \leq_{cx} Y$ is equivalent to $EX = EY$ and $Eu(X) \leq Eu(Y)$ for all increasing convex functions $u(\cdot)$. See, for example, Theorem 1.5.3 of Müller and Stoyan (2002). We have

$$EX_{t+1}(x) = E[f(\epsilon_t)x + g(\epsilon_t)] = E[f(\widehat{\epsilon}_t)x + g(\widehat{\epsilon}_t)] = E\widehat{X}_{t+1}(x).$$

Suppose now that $u(\cdot)$ is an arbitrary increasing convex function. The function $\eta(\epsilon) = f(\epsilon)x + g(\epsilon)$ is convex in ϵ for $x \geq 0$. Therefore, $\widetilde{u}(\epsilon) = u(\eta(\epsilon)) = u(f(\epsilon)x + g(\epsilon))$ is a convex function of ϵ for $x \geq 0$. Hence, for $x \geq 0$ we have

$$Eu(X_{t+1}(x)) = E\widetilde{u}(\epsilon_t) \leq E\widetilde{u}(\widehat{\epsilon}_t) = Eu(\widehat{X}_{t+1}(x)).$$

Thus, $EX_{t+1}(x) = E\widehat{X}_{t+1}(x)$ and $Eu(X_{t+1}(x)) \leq Eu(\widehat{X}_{t+1}(x))$ for any increasing and convex function $u(\cdot)$ when $x \geq 0$. This implies that $X_{t+1}(x) \leq_{cx} \widehat{X}_{t+1}(x)$ for $x \geq 0$.

Next we show that if X_t or $\widehat{X}_t$ is nonnegative a.s. for $t = 1, \ldots, T$, then $X_t \leq_{cx} \widehat{X}_t$ for $t = 2, \ldots, T$. Suppose $X_t \leq_{cx} \widehat{X}_t$. Let $u(\cdot)$ be an arbitrary convex function and let $\kappa(x) = Eu(X_{t+1}(x))$ and $\widehat{\kappa}(x) = Eu(\widehat{X}_{t+1}(x))$. Then $\kappa(x) \leq \widehat{\kappa}(x)$ for $x \geq 0$ because $u(\cdot)$ is convex and because we have already shown that $X_{t+1}(x) \leq_{cx} \widehat{X}_{t+1}(x)$ for $x \geq 0$. Moreover, $\kappa(x) = Eu(f(\epsilon_t)x + g(\epsilon_t))$ and $\widehat{\kappa}(x) = Eu(f(\widehat{\epsilon}_t)x + g(\widehat{\epsilon}_t))$ are convex in x. Therefore, if X_t is nonnegative a.s., we have

$$Eu(X_{t+1}) = E\kappa(X_t) \leq E\widehat{\kappa}(X_t) \leq E\widehat{\kappa}(\widehat{X}_t) = Eu(\widehat{X}_{t+1}).$$

If $\widehat{X}_t$ is nonnegative a.s., we have

$$Eu(X_{t+1}) = E\kappa(X_t) \leq E\kappa(\widehat{X}_t) \leq E\widehat{\kappa}(\widehat{X}_t) = Eu(\widehat{X}_{t+1}).$$

Thus, we have $X_{t+1} \leq_{cx} \widehat{X}_{t+1}$.

The proofs of (c) and (d) are similar to the proof of (b) and are omitted. □

Property (a) of the above lemma implies that for systems with the same sequence $\{\epsilon_t\}$, higher variability of the price in the first period will lead to higher variability of prices in all subsequent periods. In property (b) and property (c), $Ef(\epsilon_t) = Ef(\widehat{\epsilon}_t)$ and $Eg(\epsilon_t) = Eg(\widehat{\epsilon}_t)$ together imply that $E[X_{t+1}|X_t = x] = E[\widehat{X}_{t+1}|\widehat{X}_t = x]$, which is a necessary condition for $X_{t+1}(x) \leq_{cx} \widehat{X}_{t+1}(x)$. Properties (b), (c), and (d) state that under some conditions, if the random influence ϵ_t or $f(\epsilon_t)$ on the prices becomes more variable in some period t, then the prices will also become more variable in all subsequent periods. We next provide a few specific examples for which the preceding results tell us that the greater input price variability leads to lower expected costs.

- $\{X_t\}$ and $\{\widehat{X}_t\}$ are both i.i.d. price sequences: Suppose that $X_1 \leq_{cx} \widehat{X}_1$. To place this setting in our framework, we may take $f(\epsilon) = 0$, $g(\epsilon) = \epsilon$, and $\{\epsilon_t\}$ i.i.d. [respectively, $\{\widehat{\epsilon}_t\}$ i.i.d.] with the same distribution as X_1 [resp., $\widehat{X}_1$]. In this case, the conditions in (b) and (c) of Proposition 3 hold and hence we have that the system with more-variable input prices $\{\widehat{X}_t\}$ has lower expected costs as indicated by Theorem 1 and Corollary 1.

- $\{X_t\}$ and $\{\widehat{X}_t\}$ are both stationary AR(1) price sequences: Let $\mu = c/(1-\rho)$ for constants c and $\rho \in (-1, 1)$. Suppose that $X_{t+1} = \rho X_t + \epsilon_t + c$ where $\{\epsilon_t\}$ are i.i.d. $N(0, \sigma^2)$ and

$X_1 \sim N(\mu, \sigma^2/(1-\rho^2))$ and $\widehat{X}_{t+1} = \rho \widehat{X}_t + \widehat{\epsilon}_t + c$ where $\{\widehat{\epsilon}_t\}$ are i.i.d. $N(0, \widehat{\sigma}^2)$ and $\widehat{X}_1 \sim N(\mu, \widehat{\sigma}^2/(1-\rho^2))$. Suppose that $\sigma \le \widehat{\sigma}$. For normal random variables, $X \le_{cx} Y$ is equivalent to $EX = EY$ and $\text{Var}(X) \le \text{Var}(Y)$. Therefore, $\epsilon_t \le_{cx} \widehat{\epsilon}_t$ and $X_t \le_{cx} \widehat{X}_t$ for $t = 1, \ldots, T$. To place this setting in our framework, we may take $f(\epsilon) = \rho$ and $g(\epsilon) = \epsilon + c$. The conditions in (c) of Proposition 3 hold and hence the system with more-variable input prices again has lower expected costs.

- $\{X_t\}$ and $\{\widehat{X}_t\}$ are both (discrete-time) geometric Brownian motions: Suppose that $X_1 = \widehat{X}_1 = x$, $X_{t+1} = X_t e^{\epsilon_t}$, and $\widehat{X}_{t+1} = \widehat{X}_t e^{\widehat{\epsilon}_t}$. Suppose that $\{\epsilon_t\}$ are i.i.d. $N(\mu, \sigma^2)$ and $\{\widehat{\epsilon}_t\}$ are i.i.d. $N(\widehat{\mu}, \widehat{\sigma}^2)$ where $2\mu + \sigma^2 = 2\widehat{\mu} + \widehat{\sigma}^2$ and $\sigma \le \widehat{\sigma}$. Take $f(\epsilon) = e^\epsilon$ and $g(\epsilon) = 0$ to place this within our framework. Hence, $\{f(\epsilon_t)\}$ and $\{f(\widehat{\epsilon}_t)\}$ are i.i.d. lognormal random variables for which $Ef(\epsilon_t) = Ef(\widehat{\epsilon}_t)$ and $\text{Var}(f(\epsilon_t)) \le \text{Var}(f(\widehat{\epsilon}_t))$. Moreover, $f(\epsilon_t) \le_{cx} f(\widehat{\epsilon}_t)$; see page 63 of Müller and Stoyan (2002). Application of part (d) of Proposition 3 allows us to conclude that the system with more-variable input prices $\{\widehat{X}_t\}$ has lower expected costs.

- $\{X_t\}$ and $\{\widehat{X}_t\}$ are both Markovian martingales: Suppose that $X_1 = \widehat{X}_1 = x$, $X_{t+1} = X_t + \epsilon_t$, and $\widehat{X}_{t+1} = \widehat{X}_t + \widehat{\epsilon}_t$ where $\{\epsilon_t\}$ and $\{\widehat{\epsilon}_t\}$ are sequences of independent random variables with $E\epsilon_t = E\widehat{\epsilon}_t = 0$ and $\epsilon_t \le_{cx} \widehat{\epsilon}_t$. This example fits our framework with $f(\epsilon) = 1$ and $g(\epsilon) = \epsilon$, and part (c) of Proposition 3 allows us to apply Theorem 1 and Corollary 1 to conclude that the system with more-variable input prices has lower expected costs.

One may at first be tempted to attribute the lower costs associated with higher input price variability solely to the more frequent opportunities afforded by higher variability to place large (small) orders in periods in which prices are anticipated to be higher (lower) in subsequent periods. As we note in Section 6, this period-over-period effect is indeed important (there, for example, we observe that the relative reduction in cost due to higher variability is increasing in the length of the planning horizon and is decreasing in the correlation in prices over time). However, higher price variability yields lower expected total cost even when the input prices form a martingale (wherein the price in a current period is equal to the conditional expected price in future periods) and also when the problem has only one period.

The effect of variability can be traced to the concavity of the expected cost as a function of the input price. This concavity arises from the ability to adjust order quantities based on price realization. The order quantity in each period is determined by trading off input price, inventory holding cost, backorder cost, and expectations about future prices. The firm can benefit from lower prices by ordering more and, therefore, reducing backorder costs. Higher prices are of course harmful, but the effect is mitigated by the ability of the firm to order less and instead incur higher backorder costs. If input prices are sufficiently high, the firm stops ordering and instead incurs the backorder cost. Beyond a certain threshold, expected total cost becomes invariant to price. The above effects are easiest to see in the context of a single period problem, which we explore next.

The Single Period Case. Consider a single period version of the problem where there is only a single opportunity to order after price is revealed but before demand is realized. If demand falls below the order quantity, an overage cost per unit is incurred while if demand exceeds the order quantity, a shortage cost is incurred. To be consistent with the multi-period problem, let h denote the unit overage cost and b the unit shortage cost. Given the realized price, this is of course an instance of the classic newsvendor problem.

Let $X = X_1$ denote the random input price. Demand is denoted by D with distribution function $\Phi(\cdot)$ and density function $\phi(\cdot)$. Given price realization x, the expected total cost is

$$v(x) = \min_{y \geq 0}[xy + L(y)] = \min_{y \geq 0} w(x,y),$$

where $w(x,y) = xy + L(y)$ is the cost when price is x and the ordering quantity is y. The optimal order quantity is

$$y^*(x) = \begin{cases} \Phi^{-1}(\frac{b-x}{b+h}) & \text{if } x \leq b, \\ 0 & \text{if } x > b. \end{cases}$$

Substituting into the expression for the expected total cost leads to

$$v(x) = \begin{cases} b\int_{\Phi^{-1}(\frac{b-x}{b+h})}^{\infty} \xi\phi(\xi)d\xi - h\int_0^{\Phi^{-1}(\frac{b-x}{b+h})} \xi\phi(\xi)d\xi & \text{if } x \leq b, \\ bE[D] & \text{if } x > b, \end{cases}$$

from which we can easily show that $v(x)$ is a concave function in x. In turn, this leads to the result that, for the single period case, higher price variability leads to lower expected total cost (if $X \leq_{cx} \widehat{X}$, then $Ev(X) \geq Ev(\widehat{X})$). This is true regardless of the distribution of demand.

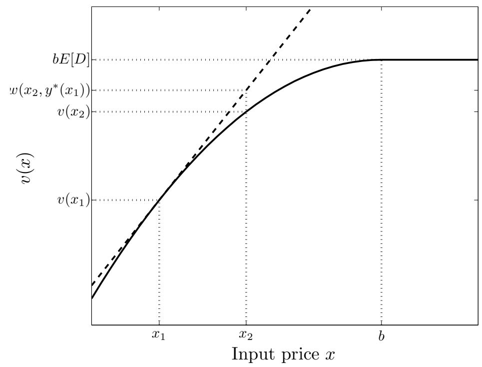


Figure 1: Cost as a function of the ordering price for stochastic demand

The concavity of the expected cost can be explained as follows. At a given price $x_1 \leq b$, the optimal order quantity is $y^*(x_1) = \Phi^{-1}(\frac{b-x_1}{b+h})$ and the associated expected cost is $x_1 y^*(x_1) + L(y^*(x_1))$. If the input price increases (decreases) from x_1 to x_2 and the order quantity is not adjusted, the expected cost would increase (decrease) linearly with rate $y^*(x_1)$ to $w(x_2, y^*(x_1)) = x_2 y^*(x_1) + L(y^*(x_1))$. However, if the order quantity is adjusted and chosen optimally, then the order quantity $y^*(x_1)$ would be lower (higher) and the optimal cost $x_2 y^*(x_2) + L(y^*(x_2))$ would be lower than that if the order quantity is not adjusted. As a consequence, the optimal expected total cost is concave in the input price. This is illustrated in Figure 1.

Note that a special case is when demand is deterministic and assumes a single value $D = d$. The optimal cost function in that case is linear with slope d for $x \leq b$ and equal to bd for $x > b$, i.e.,

$$v(x) = \begin{cases} xd & \text{if } x \leq b, \\ bd & \text{if } x > b. \end{cases}$$

If the input price is either $\mu + \alpha$ or $\mu - \alpha$ with equal probability (in which case α is the standard deviation of the input price) and $\mu \leq b$, then the optimal cost is

$$V(\alpha) = \begin{cases} \frac{1}{2}(\mu + \alpha)d + \frac{1}{2}(\mu - \alpha)d = \mu d & \text{if } \alpha \leq b - \mu, \\ \frac{1}{2}bd + \frac{1}{2}(\mu - \alpha)d = \frac{1}{2}(b + \mu - \alpha)d & \text{if } \alpha > b - \mu. \end{cases}$$

Clearly, $V(\alpha)$ is decreasing in α.

We conclude this section by noting that the benefit of input price variability is also present in other inventory systems, including systems with an infinite planning horizon, systems with lost sales instead of backorders, systems with a fixed ordering cost, and systems with fixed leadtimes. For the sake of brevity, we omit the details.

4 Impact of Price Correlation over Time

In this section, we study the impact of price correlation over time on the optimal expected total cost. To do so, we compare two different inventory systems that are identical except that they have different stationary AR(1) ordering price sequences $\{X_t\}$ and $\{\widehat{X}_t\}$ such that $X_1, \widehat{X}_1 \sim N(\mu, \sigma^2)$,

$$X_{t+1} = (1-\rho)\mu + \rho X_t + \sqrt{1-\rho^2}\epsilon_t, \tag{5}$$

$$\widehat{X}_{t+1} = (1-\widehat{\rho})\mu + \widehat{\rho}\widehat{X}_t + \sqrt{1-\widehat{\rho}^2}\epsilon_t \tag{6}$$

for $t = 1, \ldots, T-1$, and $\{\epsilon_t\}$ are i.i.d. normal random variables with mean 0 and variance σ^2 that are independent of X_1 and $\widehat{X}_1$. It is easy to check that $X_t, \widehat{X}_t \sim N(\mu, \sigma^2)$ and $\text{Corr}(X_t, X_{t+j}) = \rho^j$, $\text{Corr}(\widehat{X}_t, \widehat{X}_{t+j}) = \widehat{\rho}^j$ for $j \geq 0$. It will be helpful to view the two price sequences as random vectors, which we denote by $\mathbf{X} = (X_1, \ldots, X_T)$ and $\widehat{\mathbf{X}} = (\widehat{X}_1, \ldots, \widehat{X}_T)$. As before, we suppose that the assumptions in Section 2 hold for each of the two systems viewed in isolation.

Let $V_1(s) = Ev_1(s, X_1)$ and $\widehat{V}_1(s) = E\widehat{v}_1(s, \widehat{X}_1)$ be the optimal expected total costs for the two systems, $y_t^*(s,x)$ and $\widehat{y}_t^*(s,x)$ be the optimal order-up-to levels for the two systems in period t when the inventory is s and the ordering price is x, and $y_t^\circ(x)$ and $\widehat{y}_t^\circ(x)$ be the base stock levels for the two systems in period t when the ordering price is x. Recall from Proposition 1 that $y_t^*(s,x) = \max\{s, y_t^\circ(x)\}$ and $\widehat{y}_t^*(s,x) = \max\{s, \widehat{y}_t^\circ(x)\}$. Below, we compare $V_1(s)$ and $\widehat{V}_1(s)$

In the following developments, we will use the tool of supermodular ordering of random vectors. The supermodular order is reviewed in, e.g., Müller and Stoyan (2002) and Shaked and Shanthikumar (2007). A function $u(\cdot)$ on $\mathbb{R}^T$ is said to be supermodular if $u(\mathbf{x} + \varepsilon\mathbf{e}^i + \delta\mathbf{e}^j) - u(\mathbf{x} + \varepsilon\mathbf{e}^i) - u(\mathbf{x} + \delta\mathbf{e}^j) + u(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^T$, all $i, j = 1, \ldots, T$ with $i < j$ and all $\varepsilon, \delta > 0$. A function $u(\cdot)$ is submodular if $-u(\cdot)$ is supermodular. If $u(\cdot)$ is twice differentiable then $u(\cdot)$ is supermodular if and only if $\frac{\partial^2 u}{\partial x^i \partial x^j}(\mathbf{x}) \geq 0$ for all $\mathbf{x}$ and all i, j with $i < j$. A random vector $\mathbf{X} = (X_1, \ldots, X_T)$ is said to be smaller than a random vector

$\widehat{\mathbf{X}} = (\widehat{X}_1, \ldots, \widehat{X}_T)$ in the supermodular order, written $\mathbf{X} \leq_{sm} \widehat{\mathbf{X}}$, if $Eu(\mathbf{X}) \leq Eu(\widehat{\mathbf{X}})$ for all supermodular functions $u(\cdot)$ such that the expectations exist. The condition that $\mathbf{X} \leq_{sm} \widehat{\mathbf{X}}$ can be interpreted to mean that the entries of $\widehat{\mathbf{X}}$ have greater positive dependence than do the entries of $\mathbf{X}$; see, Müller and Stoyan (2002) or Shaked and Shanthikumar (2007). If $\mathbf{X}$ and $\widehat{\mathbf{X}}$ are normally distributed random vectors, then $\mathbf{X} \leq_{sm} \widehat{\mathbf{X}}$ is equivalent to $\mathbf{X}$ and $\widehat{\mathbf{X}}$ having the same marginal distributions and $\mathrm{Corr}(X_i, X_j) \leq \mathrm{Corr}(\widehat{X}_i, \widehat{X}_j)$ for all $i \neq j$ (see Theorem 3.13.5 of Müller and Stoyan 2002). Therefore, $\mathbf{X}$ and $\widehat{\mathbf{X}}$ in (5)–(6) satisfy $\mathbf{X} \leq_{sm} \widehat{\mathbf{X}}$ if $0 \leq \rho \leq \widehat{\rho}$.

In preparation for the proof of the main result of this section, for $k = 1, \ldots, T$, consider $\mathbf{X}_k = (X_{1,k}, \ldots, X_{T,k})$, where $X_{1,k} \sim N(\mu, \sigma^2)$,

$$X_{i+1,k} = \begin{cases} (1-\widehat{\rho})\mu + \widehat{\rho} X_{i,k} + \sqrt{1-\widehat{\rho}^2}\epsilon_i & \text{for } i = 1, \ldots, k-1, \\ (1-\rho)\mu + \rho X_{i,k} + \sqrt{1-\rho^2}\epsilon_i & \text{for } i = k, \ldots, n-1, \end{cases}$$

and $\{\epsilon_t\}$ are i.i.d. normal random variables with mean 0 and variance σ^2 that are independent of $X_{1,k}$. Note that $\mathbf{X} = \mathbf{X}_1$ and $\widehat{\mathbf{X}} = \mathbf{X}_T$. Note also that $\mathbf{X}_k$ is a non-stationary AR(1) process with $X_{i,k} \sim N(\mu, \sigma^2)$ and

$$\mathrm{Corr}(X_{i,k}, X_{j,k}) = \begin{cases} \widehat{\rho}^{j-i} & \text{for } i < j \leq k, \\ \rho^{j-i} & \text{for } k \leq i < j, \\ \widehat{\rho}^{k-i}\rho^{j-k} & \text{for } i < k < j. \end{cases}$$

It is easy to check that if $0 \leq \rho \leq \widehat{\rho}$, then $\mathrm{Corr}(X_{i,k}, X_{j,k}) \leq \mathrm{Corr}(X_{i,k+1}, X_{j,k+1})$ for all $1 \leq i < j \leq T$, from which we immediately obtain the following lemma.

Lemma 3. *Suppose that* $0 \leq \rho \leq \widehat{\rho}$. *Then* $\mathbf{X}_k \leq_{sm} \mathbf{X}_{k+1}$ *and* $(X_{t,k}, X_{t+1,k}) \leq_{sm} (X_{t,k+1}, X_{t+1,k+1})$ *for all* $t = 1, \ldots, T-1$ *and* $k = 1, \ldots, T-1$.

Let $v_{t,k}(s, x)$ be the optimal expected total cost from time t onward when the inventory is s and the ordering price is x for the system with input price sequence $\mathbf{X}_k$. Then $v_{t,k}(s, x) = \min_{y \geq s} w_{t,k}(y, x) - xs$ where

$$w_{t,k}(y, x) = xy + L(y) + \beta \int_{\xi} E[v_{t+1,k}(y-\xi, X_{t+1,k}) | X_{t,k} = x]\phi(\xi)d\xi.$$

These expressions are simply equations (2) and (3) for a system with input prices $\mathbf{X}_k$. Below we also use the notation $y^*_{t,k}(s,x)$ and $y^\circ_{t,k}(x)$ for optimal order up-to-levels and base-stock levels in this system. (Note that to place the prices $\mathbf{X}_k$ into the form (1), we must allow the functions $f_t(\cdot)$ and $g_t(\cdot)$ to depend upon t. As we noted in Section 2, our results still hold for such non-homogeneous cases.) We have $v_t(s,x) = v_{t,1}(s,x)$, $\widehat{v}_t(s,x) = v_{t,T}(s,x)$, $V_1(s) = Ev_{1,1}(s, X_{1,1})$, and $\widehat{V}_1(s) = Ev_{1,T}(s, X_{1,T})$.

Lemma 4. *Suppose that $\rho, \widehat{\rho} \geq 0$. Then $v_{t,k}(s,x)$ is submodular in (s,x) for $t = 1, \ldots, T+1$ and $k = 1, \ldots, T$.*

The main result of this section is the following theorem, which describes the impact of price correlation over time on the optimal expected cost and indicates that the optimal expected cost is increasing in that correlation if the correlation is positive.

Theorem 2. *If $0 \leq \rho \leq \widehat{\rho}$, then $V_1(s) \leq \widehat{V}_1(s)$ for all s.*

Proof. We will show that for all decreasing functions $u(\cdot)$, we have

$$Ev_{t,k}(u(X_{t,k}), X_{t,k}) \leq Ev_{t,k+1}(u(X_{t,k+1}), X_{t,k+1}), \tag{7}$$

for $k = 1, \ldots, T-1$ and $t = 1, \ldots, T$. From this the theorem follows, because for a given inventory level s, we may take $u(x) = s$ to obtain $V_1(s) = Ev_{1,1}(s, X_{1,1}) \leq Ev_{1,T}(s, X_{1,T}) = \widehat{V}_1(s)$. We establish (7) by considering the cases $t \geq k+1$, $t = k$, and $t \leq k-1$ separately.

Fix k. Given a decreasing function $u(\cdot)$, define

$$\theta_{t,k}(x) = x[y^*_{t,k}(u(x), x) - u(x)] + L(y^*_{t,k}(u(x), x)).$$

To avoid a proliferation of subscripts in the remainder of the proof, let $\overline{X}_t = X_{t,k}$ and $\underline{X}_t = X_{t,k+1}$ for $t = 1, \ldots, T$. This allows us to express $v_{t,k}(u(x), x)$ as

$$v_{t,k}(u(x), x) = \theta_{t,k}(x) + \beta \int_{\xi} E[v_{t+1,k}(y^*_{t,k}(u(x), x) - \xi, \overline{X}_{t+1}) | \overline{X}_t = x] \phi(\xi) d\xi.$$

In a given period t, the expected cost from time t onward depends upon that period's realized price x and starting inventory level s, as well as the conditional distribution of future prices in periods $t+1,\ldots,T$ given the price x in period t. Likewise, the optimal base stock level in a given period t depends only on the realized price x in that period and the conditional distribution of future prices. It does not depend on the prices or distributions of the prices in the past. Therefore, we have $y^{\circ}_{t,k}(x) = y^{\circ}_{t,k+1}(x)$ and $v_{t,k}(s,x) = v_{t,k+1}(s,x)$ for $t \geq k+1$. Hence, (7) holds for $t \geq k+1$ and for any decreasing function $u(\cdot)$ because $\overline{X}_t$ and $\underline{X}_t$ have the same distribution (both are $N(\mu,\sigma^2)$) for $t \geq k+1$.

When $t = k$, we have

$$\begin{aligned} Ev_{t,k}(u(\overline{X}_t), \overline{X}_t) &\leq E[w_{t,k}(y^*_{t,k+1}(u(\overline{X}_t), \overline{X}_t), \overline{X}_t) - \overline{X}_t u(\overline{X}_t)] \\ &= E[\theta_{t,k+1}(\overline{X}_t)] + \beta \int_{\xi} E[v_{t+1,k}(y^*_{t,k+1}(u(\overline{X}_t), \overline{X}_t) - \xi, \overline{X}_{t+1})]\phi(\xi)d\xi. \end{aligned} \tag{8}$$

In the preceding, we can replace $\overline{X}_t$ by $\underline{X}_t$ in the argument of $E(\theta_{t,k+1}(\cdot))$ because both have the same distribution. For the second term in (8), note that $v_{t+1,k}(s,x) = v_{t+1,k+1}(s,x)$ because $t+1 \geq k+1$. Moreover, $v_{t+1,k}(s,x)$ is submodular in (s,x) by Lemma 4, and $y^*_{t,k+1}(u(x),x) = \max\{u(x), y^{\circ}_{t,k+1}(x)\}$ is decreasing in x by Proposition 2. Consequently, $v_{t+1,k}(y^*_{t,k+1}(u(x_t),x_t) - \xi, x_{t+1})$ is supermodular in (x_t, x_{t+1}). By Lemma 3 we also have $(\overline{X}_t, \overline{X}_{t+1}) \leq_{sm} (\underline{X}_t, \underline{X}_{t+1})$. Therefore,

$$\begin{aligned} E[v_{t+1,k}(y^*_{t,k+1}(u(\overline{X}_t), \overline{X}_t) - \xi, \overline{X}_{t+1})] &\leq E[v_{t+1,k}(y^*_{t,k+1}(u(\underline{X}_t), \underline{X}_t) - \xi, \underline{X}_{t+1})] \\ &= E[v_{t+1,k+1}(y^*_{t,k+1}(u(\underline{X}_t), \underline{X}_t) - \xi, \underline{X}_{t+1})]. \end{aligned}$$

As a consequence, by (8) we have

$$\begin{aligned} Ev_{t,k}(u(\overline{X}_t), \overline{X}_t) &\leq E[\theta_{t,k+1}(\underline{X}_t)] + \beta \int_{\xi} E[v_{t+1,k+1}(y^*_{t,k+1}(u(\underline{X}_t), \underline{X}_t) - \xi, \underline{X}_{t+1})]\phi(\xi)d\xi \\ &= Ev_{t,k+1}(u(\underline{X}_t), \underline{X}_t), \end{aligned}$$

and so (7) holds for $t = k$.

Consider some $t \leq k$. Suppose inductively that $Ev_{t,k}(u(\overline{X}_t), \overline{X}_t) \leq Ev_{t,k+1}(u(\underline{X}_t), \underline{X}_t)$ for all decreasing functions $u(\cdot)$. Consider an arbitrary decreasing function $u(\cdot)$. We have

$$\begin{aligned} Ev_{t-1,k}(u(\overline{X}_{t-1}),\overline{X}_{t-1}) &\le E[w_{t-1,k}(y^*_{t-1,k+1}(u(\overline{X}_{t-1}),\overline{X}_{t-1}),\overline{X}_{t-1}) - \overline{X}_{t-1}u(\overline{X}_{t-1})] \\ &= E[\theta_{t-1,k+1}(\overline{X}_{t-1})] + \beta\int_\xi Ev_{t,k}(y^*_{t-1,k+1}(u(\overline{X}_{t-1}),\overline{X}_{t-1}) - \xi,\overline{X}_t)\phi(\xi)d\xi. \end{aligned}$$

Observe that $(\overline{X}_{t-1},\overline{X}_t)$ are normal random variables each with mean μ and variance σ^2 and with correlation ρ. Recall that $\overline{X}_t = (1-\rho)\mu + \rho\overline{X}_{t-1} + \sqrt{1-\rho^2}\epsilon_{t-1}$. Then $\overline{X}_{t-1}$ can be written as $\overline{X}_{t-1} = \pi(\overline{X}_t,\tilde{\epsilon}_{t,k})$, where $\pi(x,\epsilon) = (1-\rho)\mu + \rho x + \sqrt{1-\rho^2}\epsilon$ and $\tilde{\epsilon}_{t,k}$ is normally distributed with mean 0 and variance σ^2 and is independent of $\overline{X}_t$. Similarly, we have $\underline{X}_{t-1} = \pi(\underline{X}_t,\tilde{\epsilon}_{t,k+1})$, where $\tilde{\epsilon}_{t,k+1}$ is normally distributed with mean 0 and variance σ^2 and is independent of $\underline{X}_t$. Note that $\tilde{\epsilon}_{t,k}$ and $\tilde{\epsilon}_{t,k+1}$ have the same distribution (which is the distribution of ϵ_t). Let $\eta(x,\epsilon) = y^*_{t-1,k+1}(u(\pi(x,\epsilon)),\pi(x,\epsilon))$. Since $\rho \ge 0$, we have that $\pi(x,\epsilon)$ is an increasing function of x. Therefore, $\eta(x,\epsilon)$ is a decreasing function of x by Proposition 2. By the inductive assumption, we have

$$Ev_{t,k}(\eta(\overline{X}_t,\epsilon)) - \xi,\overline{X}_t) \le Ev_{t,k+1}(\eta(\underline{X}_t,\epsilon)) - \xi,\underline{X}_t)$$

for any realization ϵ. As a consequence,

$$\begin{aligned} Ev_{t-1,k}(u(\overline{X}_{t-1}),\overline{X}_{t-1}) &\le E[\theta_{t-1,k+1}(\overline{X}_{t-1})]) + \beta\int_\xi\int_\epsilon Ev_{t,k}(\eta(\overline{X}_t,\epsilon) - \xi,\overline{X}_t)\Psi(d\epsilon)\phi(\xi)d\xi \\ &\le E[\theta_{t-1,k+1}(\underline{X}_{t-1})]) + \beta\int_\xi\int_\epsilon Ev_{t,k+1}(\eta(\underline{X}_t,\epsilon) - \xi,\underline{X}_t)\Psi(d\epsilon)\phi(\xi)d\xi \\ &= E[\theta_{t-1,k+1}(\underline{X}_{t-1})]) + \beta\int_\xi E[v_{t,k+1}(y^*_{t-1,k+1}(u(\underline{X}_{t-1}),\underline{X}_{t-1}) - \xi,\underline{X}_t)]\phi(\xi)d\xi \\ &= Ev_{t-1,k+1}(u(\underline{X}_{t-1}),\underline{X}_{t-1}). \end{aligned}$$

Therefore (7) holds for all $t \le k$ by induction on t. □

The preceding theorem shows that if the input prices follow a stationary AR(1) process, then greater positive price correlation over time yields larger expected total cost. This result should be intuitive. With high positive correlation in prices, an unusually high price is often followed by another unusually high price, and therefore delaying purchase will likely not avoid high costs.

Therefore, high correlation in prices over time leads to high expected total cost. On the other hand, with low positive correlation, if the price is unusually high in one period then the probability that the price in the next period will continue to be high is comparatively small, and hence purchases can be delayed in expectation of a price decrease.

5 Inventory Systems with Multiple Inputs

In this section, we extend our analysis to systems with multiple input components, where one unit of each of n input components is needed to satisfy one unit of demand. The ordering prices of these n components are stochastic (deterministic prices can be treated as a special case). The holding cost of component $i = 1, \ldots, n$ is h^i.

As in the single component model, we assume that the price X_{t+1}^i of component i in period $t+1$ is dependent on the price X_t^i of component i in period t:

$$X_{t+1}^i = f^i(\epsilon_t^i) X_t^i + g^i(\epsilon_t^i), \qquad t = 1, \ldots, T-1,$$

where $\{\boldsymbol{\epsilon}_t = (\epsilon_t^1, \ldots, \epsilon_t^n)' : t = 1, \ldots, T-1\}$ is a sequence of independent random vectors. Let $\mathbf{X}_t = (X_t^1, \ldots, X_t^n)'$ for $t = 1, \ldots, T$. We assume that $\{\boldsymbol{\epsilon}_t\}$, $\mathbf{X}_1$, and the sequence of demands are independent. The prices of different components in the same period may be correlated. Other assumptions are the same as those of the single component model.

The problem can be viewed as a Markov decision process where the state of the system at the beginning of each period is $(\mathbf{s}, \mathbf{x})$ where $\mathbf{s} = (s^1, \ldots, s^n)'$ is the vector of net inventory levels and $\mathbf{x} = (x^1, \ldots, x^n)'$ is the vector of input prices. In each period, the action, i.e., the decision to be made, is the vector of order-up-to net inventory levels $\mathbf{y} = (y^1, \ldots, y^n)'$ where $y_i \in [s_i, \infty)$ for $i = 1, \ldots, n$. If, in a particular period, we bring the net inventory up to $\mathbf{y}$, and the realized demand is ξ, then the net inventory level in the subsequent period is $\mathbf{y} - \xi$.

For a given state $(\mathbf{s}, \mathbf{x})$ at the beginning of period t, let $s^k = \min\{s^1, \ldots, s^n\}$. To compute the ordering cost and the one-period holding and shortage cost, we consider two cases: (i) $s^k \geq 0$ and (ii) $s^k < 0$. Let $\widehat{y} = \min\{y^1, \ldots, y^n\}$. In case (i), we have no backorders, and the inventory level of component i is s^i for $i = 1, \ldots, n$. If we decide to bring the net inventory level up to $\mathbf{y}$, then the ordering cost is $\sum_{i=1}^{n} x^i(y^i - s^i)$. Note that in this period we can satisfy at most $\widehat{y}$ units of demand. If demand D is less than or equal to $\widehat{y}$, the holding cost for component i is $h^i(y^i - D)$. If demand D is larger than $\widehat{y}$, the holding cost for component i is $h^i(y^i - \widehat{y})$ and the backorder cost is $b(D - \widehat{y})$. Therefore, the one-period holding and shortage cost is

$$
\begin{aligned}
L(\mathbf{y}) &= \sum_{i=1}^{n} h^i \left(\int_0^{\widehat{y}} (y^i - \xi)\phi(\xi)d\xi + \int_{\widehat{y}}^{\infty} (y^i - \widehat{y})\phi(\xi)d\xi \right) + b \int_{\widehat{y}}^{\infty} (\xi - \widehat{y})\phi(\xi)d\xi \\
&= \sum_{i=1}^{n} h^i E(\widehat{y} - D)^+ + bE(D - \widehat{y})^+ + \sum_{i=1}^{n} h^i(y^i - \widehat{y}). \qquad (9)
\end{aligned}
$$

In case (ii), we have $-s^k$ units of backorders, and the inventory level of component i is $s^i - s^k$ for $i = 1, \ldots, n$. If we decide to bring the net inventory level up to $\mathbf{y}$, (or equivalently, we decide to bring the inventory level up to $\mathbf{y} - s^k$), then the ordering cost is

$$
\sum_{i}^{n} x^i[(y^i - s^k) - (s^i - s^k)] = \sum_{i=1}^{n} x^i(y^i - s^i).
$$

After bringing the inventory levels to $\mathbf{y} - s^k$, we can satisfy at most $\min\{y^1 - s^k, y^2 - s^k, \ldots, y^n - s^k\} = \widehat{y} - s^k$ units of backorders and new demand. Backorders and new demand combined equal $D - s_k$. So, by the same argument that gave us (9), the one-period holding and shortage cost is

$$
\begin{aligned}
\tilde{L}(\mathbf{y}, \mathbf{s}) &= \sum_{i=1}^{n} h^i E[(\widehat{y} - s^k) - (D - s^k)]^+ + bE[(D - s^k) - (\widehat{y} - s^k)]^+ + \sum_{i=1}^{n} h^i[(y^i - s^k) - (\widehat{y} - s^k)] \\
&= \sum_{i=1}^{n} h^i E(\widehat{y} - D)^+ + bE(D - \widehat{y})^+ + \sum_{i=1}^{n} h^i(y^i - \widehat{y}) \\
&= L(\mathbf{y}).
\end{aligned}
$$

For both cases, the ordering cost is $\sum_{i=1}^{n} x^i(y^i - s^i)$ and the one-period holding and shortage cost is

$$
L(\mathbf{y}) = \sum_{i=1}^{n} h^i \left(\int_0^{\widehat{y}} (y^i - \xi)\phi(\xi)d\xi + \int_{\widehat{y}}^{\infty} (y^i - \widehat{y})\phi(\xi)d\xi \right) + b \int_{\widehat{y}}^{\infty} (\xi - \widehat{y})\phi(\xi)d\xi,
$$

where $\widehat{y} = \min\{y^1, \ldots, y^n\}$.

Let $\mathbf{g}(\boldsymbol{\epsilon}_t) = (g^1(\epsilon_t^1), \ldots, g^n(\epsilon_t^n))'$ and let $\Lambda(\boldsymbol{\epsilon}_t)$ be the $n \times n$ matrix with diagonal entries

$f^1(\epsilon_t^1), \ldots, f^n(\epsilon_t^n)$ and other entries 0. Then we have $\mathbf{X}_{t+1} = \Lambda(\boldsymbol{\epsilon}_t)\mathbf{X_t} + \mathbf{g}(\boldsymbol{\epsilon}_t)$. The optimality equations are

$$\begin{aligned} v_t(\mathbf{s}, \mathbf{x}) &= \min_{\mathbf{y} \geq \mathbf{s}} \left\{ \mathbf{x}'(\mathbf{y} - \mathbf{s}) + L(\mathbf{y}) + \beta \int_\xi E[v_{t+1}(\mathbf{y} - \xi, \mathbf{X_{t+1}}) | \mathbf{X} = \mathbf{x}] \phi(\xi) d\xi \right\} \\ &= \min_{\mathbf{y} \geq \mathbf{s}} \left\{ \mathbf{x}'\mathbf{y} + L(\mathbf{y}) + \beta \int_\xi E v_{t+1}(\mathbf{y} - \xi, \Lambda(\boldsymbol{\epsilon}_t)\mathbf{x} + \mathbf{g}(\boldsymbol{\epsilon}_t)) \phi(\xi) d\xi \right\} - \mathbf{x}'\mathbf{s} \\ &= \min_{\mathbf{y} \geq \mathbf{s}} w_t(\mathbf{y}, \mathbf{x}) - \mathbf{x}'\mathbf{s} \end{aligned}$$

and $v_{T+1}(\mathbf{s}, \mathbf{x}) = 0$, where

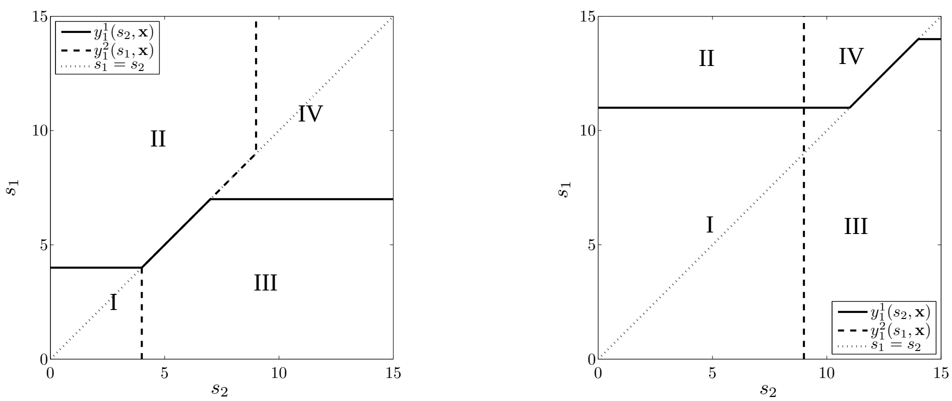
$$w_t(\mathbf{y}, \mathbf{x}) = \mathbf{x}'\mathbf{y} + L(\mathbf{y}) + \beta \int_\xi E v_{t+1}(\mathbf{y} - \xi, \Lambda(\boldsymbol{\epsilon}_t)\mathbf{x} + \mathbf{g}(\boldsymbol{\epsilon}_t)) \phi(\xi) d\xi.$$

Lemma 5. *$L(\mathbf{y})$ is a convex and submodular function of $\mathbf{y}$ and $v_t(\mathbf{s}, \mathbf{x})$ is a convex and submodular function of $\mathbf{s}$ for all $\mathbf{x}$ and $t = 1, \ldots, T+1$.*

The following theorem follows directly from Lemma 5.

Theorem 3. *The optimal policy is a state-dependent base stock policy for each component. For component i, there exists a base stock level $y_t^i(\mathbf{s}^{-i}, \mathbf{x})$ where $\mathbf{s}^{-i} = (s^1, \ldots, s^{i-1}, s^{i+1}, \ldots, s^n)$ such that if the starting net inventory s^i in period t is less than $y_t^i(\mathbf{s}^{-i}, \mathbf{x})$, then we order up to $y_t^i(\mathbf{s}^{-i}, \mathbf{x})$; otherwise, we do not order. That is, the optimal order-up-to level for component i in state $(\mathbf{s}, \mathbf{x})$ is $\max\{s^i, y_t^i(\mathbf{s}^{-i}, \mathbf{x})\}$. In addition, the base stock level $y_t^i(\mathbf{s}^{-i}, \mathbf{x})$ is increasing in each s_j for $j \neq i$.*

The structure of the optimal policy is illustrated in Figure 2 for a system with two components, where Figures 2a and 2b illustrate the policy in period 1 for two different realized prices. When the starting inventory for the two components is in region I, we order both components; in region II, we order only component 2; in region III, we order only component 1; and in region IV, we order nothing. The figure provides some insights into the effect of the price of component 1 (the price of component 2 is fixed in this example). First, notice that a decrease in the price of component 1 leads to higher order up to levels for both components 1 and 2. Second, notice that the optimal policy may not always seek to balance the inventory of both components. For example, when the starting inventory is in region I and the price is high, it is optimal to balance the inventory of the two components. However, when the price is low, it is optimal to bring the inventory of component 1 to a higher level than that of component 2 to take advantage of the lower price of component 1 (more of component 2 can always be ordered in future periods at the same price).



(a) $X_1^1 = 70$

(b) $X_1^1 = 30$

Figure 2: Structure of the optimal policy for different realizations of input prices. In this example, the price of component 1 is stochastic and the price of component 2 is fixed. Demand is uniformly distributed on [1,15], $T = 12$, $P(X_t^1 = 30) = P(X_t^1 = 70) = 0.5$ for all t, $X_t^2 = 40$ for all t, $\beta = 0.99$, $b = 50$, $h_1 = 20$, and $h_2 = 40$.

We provide conditions under which the base stock levels are decreasing with respect to the realized price of each component in the following proposition.

Proposition 4. *If* $0 \leq f^i(\epsilon) \leq 1$ *for all* ϵ *and* $i = 1, \ldots, n$*, then* $y_t^i(\mathbf{s}^{-i}, \mathbf{x})$ *is decreasing in each* x_j *for* $j = 1, \ldots, n$*.*

The base stock level $y_t^i(\mathbf{s}^{-i}, \mathbf{x})$ need not be decreasing in x_j if the condition in the above proposition is not satisfied. If $f^i(\epsilon) > 1$ for some i, it is possible that a high (low) price of component i in one period would lead to an even higher (lower) expected price of component i in the next period, and it may be optimal to order more (less) of component i when the price of component i is high (low). If $f^i(\epsilon) < 0$ for some i, an increase in the price of component i would lead to a decrease in the expected price of component i in the next period and possibly an increase in the order up to level for component i in the next period. To keep up with a higher order up to level of component i in the next period, it may be optimal to order more of other components. Therefore, in this case, the order up to level for the other components may be increasing in the price of component i.

Impact of Price Variability. With regard to the impact of price variability on the optimal expected total cost, we have similar results as in the single component case. Consider two different inventory systems with input price sequences $\{\mathbf{X}_t\}$ and $\{\widehat{\mathbf{X}}_t\}$ satisfying $X^i_{t+1} = f^i(\epsilon^i_t)X^i_t + g^i(\epsilon^i_t)$ and $\widehat{X}^i_{t+1} = f^i(\widehat{\epsilon}^i_t)\widehat{X}^i_t + g^i(\widehat{\epsilon}^i_t)$. All other parameters of the two systems are the same. Let $v_t(\mathbf{s}, \mathbf{x})$ and $\widehat{v}_t(\mathbf{s}, \mathbf{x})$ be the optimal total cost-to-go in period t when the net inventory levels are $\mathbf{s}$ and the input prices are $\mathbf{x}$ in period t for the two systems.

The following theorem shows that higher variability in the input prices yields lower optimal expected total cost. Here we use the notion of convex orders of random vectors. A random vector $\mathbf{X}$ is said to be smaller than $\widehat{\mathbf{X}}$ in the *convex order* (written $\mathbf{X} \leq_{cx} \widehat{\mathbf{X}}$) if $Eu(\mathbf{X}) \leq Eu(\widehat{\mathbf{X}})$ for all convex functions $u(\cdot)$ such that the expectations exist. The convex order of random vectors is reviewed in, for example, Müller and Stoyan (2002) and Shaked and Shanthikumar (2007). If $\mathbf{X} = (X^1, \ldots, X^n)$ and $\widehat{\mathbf{X}} = (\widehat{X}^1, \ldots, \widehat{X}^n)$ each have independent components, then $\mathbf{X} \leq_{cx} \widehat{\mathbf{X}}$ is equivalent to $X^i \leq_{cx} \widehat{X}^i$ for all $i = 1, \ldots, n$. If $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\widehat{\mathbf{X}} \sim N(\widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}})$, then $\mathbf{X} \leq_{cx} \widehat{\mathbf{X}}$ if and only if $\boldsymbol{\mu} = \widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}$ is positive semidefinite.

Theorem 4. *Suppose* $\mathbf{X}_1 \leq_{cx} \widehat{\mathbf{X}}_1$. *If*

(a) $\boldsymbol{\epsilon}_t = \widehat{\boldsymbol{\epsilon}}_t$ *for* $t = 1, \ldots, T$, *or*

(b) $\boldsymbol{\epsilon}_t \leq_{cx} \widehat{\boldsymbol{\epsilon}}_t$ *for* $t = 1, \ldots, T$ *and* $f^i(\cdot)$ *and* $g^i(\cdot)$ *are affine functions for* $i = 1, \ldots, n$, *or*

(c) $(f^1(\epsilon^1_t), \ldots, f^n(\epsilon^n_t)) \leq_{cx} (f^1(\widehat{\epsilon}^1_t), \ldots, f^n(\widehat{\epsilon}^n_t))$ *for* $t = 1, \ldots, T$ *and* $g^i(\cdot)$ *is a constant for* $i = 1, \ldots, n$,

then

(1) $v_t(\mathbf{s}, \mathbf{x}) \geq \widehat{v}_t(\mathbf{s}, \mathbf{x})$ *for all* $\mathbf{s}$, $\mathbf{x}$ *and* $t = 1, \ldots, T$;

(2) $E[v_t(\mathbf{s}, \mathbf{X}_t)|\mathbf{X}_{t-1} = \mathbf{x}] \geq E[\widehat{v}_t(\mathbf{s}, \widehat{\mathbf{X}}_t)|\widehat{\mathbf{X}}_{t-1} = \mathbf{x}]$ *for all* $\mathbf{s}$, $\mathbf{x}$ *and* $t = 2, \ldots, T$; *and*

(3) $Ev_t(\mathbf{s}, \mathbf{X}_t) \geq E\widehat{v}_t(\mathbf{s}, \widehat{\mathbf{X}}_t)$ *for all* $\mathbf{s}$ *and* $t = 1, \ldots, T$.

Impact of Correlation across Component Prices. Next, we study the impact of correlation across component prices on the optimal expected total cost. We compare the expected costs of two different inventory systems where the correlations across component prices in one system are larger than those in the other system in every period. More precisely, we consider two systems such that for each time t, the random input price vectors $\mathbf{X}_t$ and $\widehat{\mathbf{X}}_t$ of the two systems are comparable in the supermodular order; i.e., $\mathbf{X}_t \leq_{sm} \widehat{\mathbf{X}}_t$. Recall that this implies that the price correlations are ordered as well; i.e., $\mathrm{Corr}(X_t^i, X_t^j) \leq \mathrm{Corr}(\widehat{X}_t^i, \widehat{X}_t^j)$ for all i, j.

Let $\mathbf{X}_{t+1}(\mathbf{x}) = \Lambda(\boldsymbol{\epsilon}_t)\mathbf{x} + \mathbf{g}(\boldsymbol{\epsilon}_t)$ be a random vector that follows the conditional distribution of $\mathbf{X}_{t+1}$ given $\mathbf{X}_t = \mathbf{x}$ and let $\widehat{\mathbf{X}}_{t+1}(\mathbf{x}) = \Lambda(\widehat{\boldsymbol{\epsilon}}_t)\mathbf{x} + \mathbf{g}(\widehat{\boldsymbol{\epsilon}}_t)$ be a random vector that follows the conditional distribution of $\widehat{\mathbf{X}}_{t+1}$ given $\widehat{\mathbf{X}}_t = \mathbf{x}$.

Lemma 6. *Consider two price sequences $\{X_t^i\}$ and $\{\widehat{X}_t^i\}$, where $X_{t+1}^i = f^i(\epsilon_t^i)X_t^i + g^i(\epsilon_t^i)$ and $\widehat{X}_{t+1}^i = f^i(\widehat{\epsilon}_t^i)\widehat{X}_t^i + g^i(\widehat{\epsilon}_t^i)$. If $\mathbf{X}_1 \leq_{sm} \widehat{\mathbf{X}}_1$, $f^i(\epsilon^i)f^j(\epsilon^j) \geq 0$ for all ϵ^i, ϵ^j, $i \neq j$, and either:*

(a) $\boldsymbol{\epsilon}_t = \widehat{\boldsymbol{\epsilon}}_t$ for $t = 1, \ldots, T$ or

(b) $\boldsymbol{\epsilon}_t \leq_{sm} \widehat{\boldsymbol{\epsilon}}_t$ for $t = 1, \ldots, T$, $f^i(\cdot)$ is a constant for $i = 1, \ldots, n$, and $g^i(\cdot)$ is either increasing for all $i = 1, \ldots, n$ or decreasing for all $i = 1, \ldots, n$,

then $\mathbf{X}_t \leq_{sm} \widehat{\mathbf{X}}_t$ and $\mathbf{X}_t(\mathbf{x}) \leq_{sm} \widehat{\mathbf{X}}_t(\mathbf{x})$ for all $\mathbf{x}$ and $t = 1, \ldots, T$.

One example of property (b) is the case where each component price evolves according to an AR(1) process; i.e., $X_{t+1}^i = \rho^i X_t^i + \epsilon_t^i + c^i$ where $\rho^i \rho^j \geq 0$ for all $i \neq j$.

Lemma 7. *If $f^i(\epsilon^i)f^j(\epsilon^j) \geq 0$ for all ϵ^i, ϵ^j, $i \neq j$, then $v_t(\mathbf{s}, \mathbf{x})$ is a submodular function of $\mathbf{x}$ for all $\mathbf{s}$ and $t = 1, \ldots, T+1$.*

From the definition of the supermodular order, we have the following theorem describing the impact of correlation over component prices on the optimal expected total cost.

Theorem 5. *Suppose the conditions in Lemma 6 hold. Then*

(1) $v_t(\mathbf{s}, \mathbf{x}) \leq \widehat{v}_t(\mathbf{s}, \mathbf{x})$ for all $\mathbf{s}$, $\mathbf{x}$ and $t = 1, \ldots, T$;

(2) $E[v_t(\mathbf{s}, \mathbf{X}_t)|\mathbf{X}_{t-1} = \mathbf{x}] \leq E[\widehat{v}_t(\mathbf{s}, \widehat{\mathbf{X}}_t)|\widehat{\mathbf{X}}_{t-1} = \mathbf{x}]$ for all $\mathbf{s}$, $\mathbf{x}$ and $t = 2, \ldots, T$; and

(3) $Ev_t(\mathbf{s}, \mathbf{X}_t) \leq E\widehat{v}_t(\mathbf{s}, \widehat{\mathbf{X}}_t)$ for all $\mathbf{s}$ and $t = 1, \ldots, T$.

Proof. It is easy to check by backward induction that $v_t(\mathbf{s}, \mathbf{x}) \leq \widehat{v}_t(\mathbf{s}, \mathbf{x})$ for all $\mathbf{s}$ and $\mathbf{x}$. By Lemma 6, we have $\mathbf{X}_t(\mathbf{x}) \leq_{sm} \widehat{\mathbf{X}}_t(\mathbf{x})$ and $\mathbf{X}_t \leq_{sm} \widehat{\mathbf{X}}_t$ for $\mathbf{x}$ and $t = 1, \ldots, T$. Therefore,

$$\begin{aligned} E[v_t(\mathbf{s}, \mathbf{X}_t)|\mathbf{X}_{t-1} = \mathbf{x}] &= Ev_t(\mathbf{s}, \mathbf{X}_t(\mathbf{x})) \\ &\leq E\widehat{v}_t(\mathbf{s}, \mathbf{X}_t(\mathbf{x})) \\ &\leq E\widehat{v}_t(\mathbf{s}, \widehat{\mathbf{X}}_t(\mathbf{x})) \\ &= E[\widehat{v}_t(\mathbf{s}, \widehat{\mathbf{X}}_t)|\widehat{\mathbf{X}}_{t-1} = \mathbf{x}], \end{aligned}$$

where the second inequality follows from Lemma 7. Similarly, $Ev_t(\mathbf{s}, \mathbf{X}_t) \leq E\widehat{v}_t(\mathbf{s}, \mathbf{X}_t) \leq E\widehat{v}_t(\mathbf{s}, \widehat{\mathbf{X}}_t)$. ☐

This theorem shows that expected total cost is decreasing in correlation across component prices, implying that higher correlation across component prices is beneficial. To provide some intuition as to why such higher correlation leads to lower costs, consider the single period case. In that case, it is optimal to always order the same quantity of each component (assuming equal starting inventory levels). Therefore, the problem reduces to one of a single component with unit price equal to the sum of the unit prices of all the components. If we let X be the random variable that describes this "equivalent" unit price and $X^1, \ldots, X^n$ be the individual component prices, then $X = X^1 + X^2 + \cdots + X^n$ and $\text{Var}(X) = \sum_{i=1}^{n} \text{Var}(X^i) + \sum_{i \neq j} \text{Cov}(X^i, X^j)$. As we can see, higher price correlation leads to higher price variance, which for several common distributions, also implies higher price variability as measured by the convex order. Therefore, in such cases, higher correlation would lead to lower cost. Correlation can also impact cost by affecting order up to levels. For example, in the settings described by Proposition 4, the order up to level of each component is decreasing in the price of all other components. Therefore, when the price of one component is low and it is desirable to order more, it is preferable that the prices of other components are also low. Otherwise, the opportunity to take advantage of price variability is diminished.

6 Numerical Results

In this section, we provide numerical results that illustrate how the relative benefits of price variability, price correlation over time, and price correlation over components are affected by various problem parameters. First, we compare the performance of systems with and without price variability and compare the performance of systems with and without correlation over periods for systems with a single component. Then for systems with multiple input components, we compare the performance of systems with and without correlation across component prices.

Let $v_t(s,x)$ be the optimal cost-to-go in period t when the beginning inventory is s and the realization of price is x in period t for the system with input price sequence $\{X_t\}$ and let $\bar{v}_t(s)$ be the optimal cost-to-go in period t when the beginning inventory is s for the system with fixed input price sequence $\{\mu_t\}$, where $\mu_t = EX_t$. The relative benefit of price variability, which we denote by $\delta_{\rm v}$, is defined as follows:

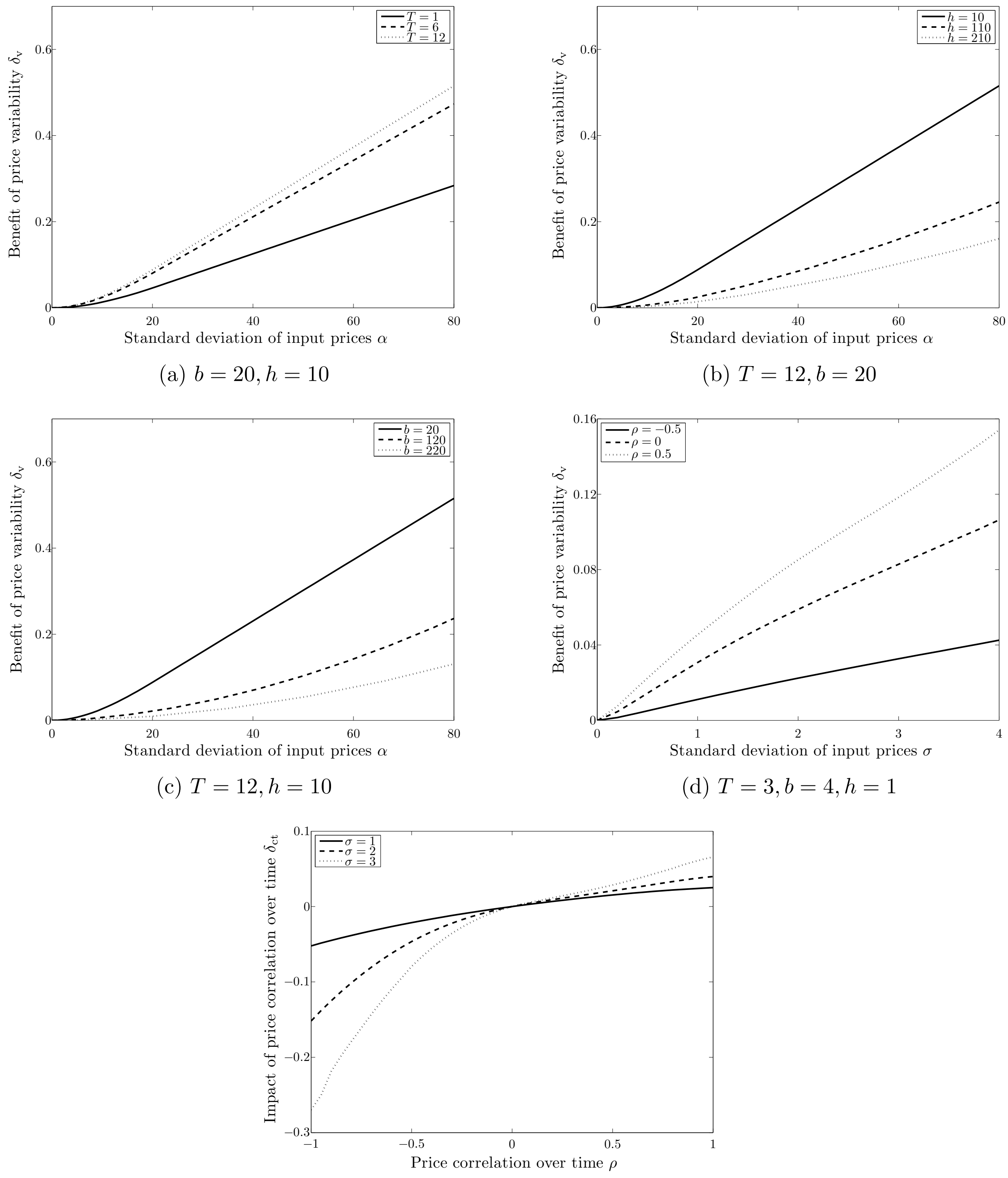
$$\delta_{\rm v} = \frac{\bar{v}_1(s) - Ev_1(s, X_1)}{\bar{v}_1(s)}.$$

In Figures 3a–3d, we examine the relative benefit of price variability for different lengths of planning horizons, different holding costs, different backorder costs, and different levels of price correlation over time. In all the numerical examples, we set the initial inventory to be 0, the discount factor to be $\beta = 0.99$, and demand to be uniformly distributed on [1,30]. (Results are qualitatively the same for other common distributions we tested.) In Figures 3a–3c, the input prices are i.i.d. across periods with $P(X_t = 80 + \alpha) = P(X_t = 80 - \alpha) = 0.5$, in which case the standard deviation of input prices is α. In Figure 3d, the input prices follow a stationary AR(1) process, namely, $X_{t+1} = (1-\rho)\mu + \rho X_t + \sqrt{1-\rho^2}\epsilon_t$, where $\mu = EX_1 = 6$, $\{\epsilon_t\}$ are normally distributed with mean 0 and variance σ^2, and $\sigma^2 = \mathrm{Var}(X_1)$.

In Figure 3e, we examine the relative impact of price correlation over time for different levels of price variability. Here, the sequence of input prices $\{X_t\}$ is a stationary AR(1) process with mean $\mu = 6$. Let $v_t(s,x,\rho)$ be the optimal cost-to-go in period t when the beginning inventory is s and the realization of price is x in period t for the above system. Note that $v_t(s,x,\rho)$ is increasing in $\rho \geq 0$ by Theorem 2. The relative impact of price correlation over time, which denoted by $\delta_{\rm ct}$, is defined as follows:

$$\delta_{\rm ct} = \frac{Ev_1(s, X_1, \rho) - Ev_1(s, X_1, 0)}{Ev_1(s, X_1, 0)}.$$

For inventory systems with multiple inputs, we examine how the benefit of price variability is affected by price correlation across components and how the relative impact of price correlation across components is affected by other parameters. We consider a 2-component, 2-period problem



(a) $b = 20, h = 10$

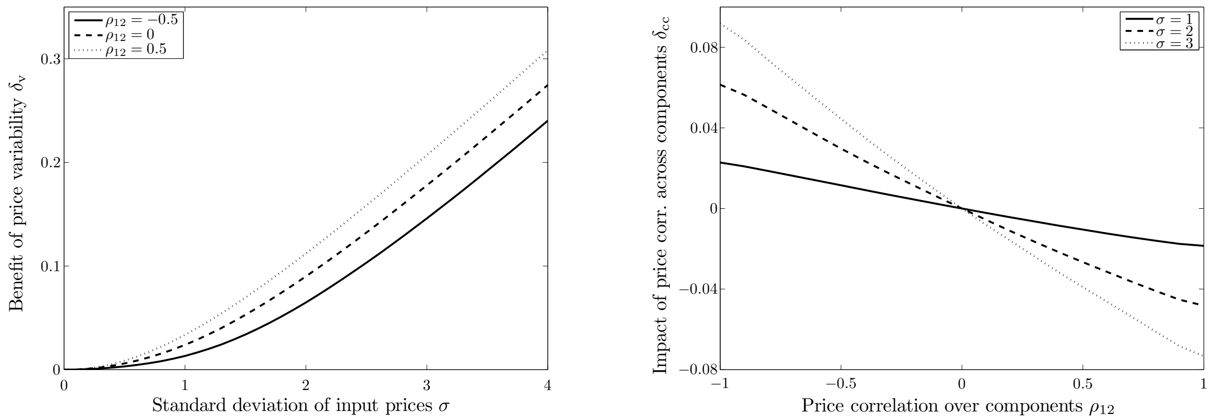
(b) $T = 12, b = 20$

(c) $T = 12, h = 10$

(d) $T = 3, b = 4, h = 1$

(e) $T = 3,\ b = 4, h = 1$

Figure 3: Impact of price variability and price correlation over time in systems with a single input.



(a) $b = 2, h_1 = h_2 = 2$ (b) $b = 2, h_1 = h_2 = 2$

Figure 4: Impact of price variability and correlation across components in systems with multiple inputs.

as an example. The input prices of the components are i.i.d. and in each period the prices are normally distributed with mean $\boldsymbol{\mu}$ and variance matrix $\boldsymbol{\Sigma}$, where $\mu_1 = \mu_2 = 6$, $\sigma_{11} = \sigma_{22}$, and $\sigma_{12} = \sigma_{21} = \sigma_{11}\rho_{12}$. Let $v_t(\mathbf{s}, \mathbf{x}, \rho_{12})$ be the optimal cost-to-go in period t when the beginning inventory levels are $\mathbf{s}$, the realizations of prices are $\mathbf{x}$ and the price correlation between the two components is ρ_{12} in period t. Then the relative impact of price correlation across components, which is denoted by δ_{cc}, is defined as follows:

$$\delta_{\text{cc}} = \frac{Ev_1(\mathbf{s}, \mathbf{X}, \rho_{12}) - Ev_1(\mathbf{s}, \mathbf{X}, 0)}{Ev_1(\mathbf{s}, \mathbf{X}, 0)}.$$

Figure 4a shows the relative benefit of price variability for different levels of component price correlation and Figure 4b shows the relative impact of component price correlation for different levels of price variability.

Based on Figure 3 and Figure 4, we can make the following observations (we provide some intuition to explain these observations; however, we caution that, in general, the interactions between various factors can be quite complex).

Observation 1: *The relative benefit of price variability is increasing in the length of the planning horizon.* This is illustrated in Figure 3a. When prices are high, a firm can order less and take advantage of the possibility of backordering and fulfilling demand in future periods. Similarly, when prices are low, a firm can order more and take advantage of the possibility of holding inventory and using this inventory to fulfill demand in future periods. The advantage derived from the flexibility of either backordering or carrying inventory across periods (to which we refer as the period-over-period effect) increases with the length of the planning horizon, as the opportunity to exercise this flexibility also increases.

Observation 2: *The relative benefit of price variability is decreasing in the holding and backorder costs.* This is illustrated in Figures 3b and 3c. When either the holding or the backorder cost is high, taking advantage of the period-over-period effect (ordering more and holding inventory or ordering less and backordering) becomes less desirable. In turn, this diminishes the benefit that may be derived from higher price variability.

Observation 3: *The relative benefit of price variability is decreasing in the price correlation over time.* This is illustrated in Figure 3d. The benefit derived from ordering more (less) in periods when prices are low (high) diminishes with correlation over time, as a low (high) price period tends to be followed by another low (high) price period.

Observation 4: *The relative benefit from lower correlation is increasing in price variability.* This is illustrated in Figure 3e. Lower correlation provides an opportunity to take advantage of the period-over-period effect. This opportunity is enhanced when price variability is high. We can also see from Figure 3e that for the examples depicted there, systems with uncorrelated prices have greater expected cost than systems with negatively correlated prices. This suggests that we can perhaps relax the condition that correlations are positive in Theorem 2, at least in some cases.

Observation 5: *The relative benefit of price variability is increasing in the price correlation across components.* This is illustrated in Figure 4a. In general, the interaction between price variability and price correlation across components is complex and depends on the correlations of prices of components over time. However, higher price correlation among components typically enables a firm to take better advantage of variability. For example, when the price of a component is low, we may prefer to buy more of that component. The value of doing so is greater when we also prefer to buy more of other components (recall that one unit of each component is needed to fulfill demand). Such opportunities will arise more frequently when prices across components are more correlated than when they are less so.

Observation 6: *The relative benefit of higher price correlation across components is increasing in price variability.* This is illustrated in Figure 4b. Higher correlation typically implies that when it is preferable to order more (less) of one component it is also preferable to order more (less) of other components. This matching of inventory levels across components is more valuable when the price variability of components is high and the benefit from adjusting order quantities is greater.

7 Conclusion

In this paper, we examined the impact of input price variability on expected cost in inventory systems with stochastic demand and stochastic input prices. For a general class of such systems, we showed that higher input price variability leads to lower expected cost. We showed that this is true for a wide range of assumptions regarding price evolution, including i.i.d. prices and prices that evolve according to a Markovian martingale. We also showed that this is true for systems with both single and multiple periods. We described how the impact of price variability on expected cost can be traced to the concavity of the cost function in input price, which is itself a consequence of the flexibility in adjusting the order quantity as prices vary. In addition, we examined the impact of price correlation over time and across inputs. We found that expected cost is increasing in price correlation over time and decreasing in price correlation across components. Numerical results suggest that higher correlation of prices over time diminishes the benefit derived from price variability while higher correlation of prices across components enhances it.

There are several avenues for future research. It would be useful to extend the analysis to broader classes of systems, including systems with multiple production stages where different components may be needed at different stages. In particular, it would be of interest to investigate how the position of a component in the production process affects the benefit derived from the variability in its input price (e.g., is price variability more beneficial for components that are upstream in the production process or is it more so for components that are downstream?). It would also be useful to consider settings in which there is variability in both the input purchase price and the output selling price. For example, a firm may purchase input from one spot market and sell output to another, with the firm observing both input and output prices at the beginning of each period and then deciding on how much input to buy and how much output to produce and sell. Lastly, it would be valuable to extend our analysis to settings where the firm may not be risk neutral and to account for its attitude toward risk by studying a decision criterion other than expected value.

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Appendix

Proof of Lemma 1: Observe first that $w_T(y,x) = xy + L(y)$ is convex in y for all x. Consider arbitrary $t \in \{1, \ldots, T-1\}$ and suppose inductively that $\frac{\partial^2 w_{t+1}}{\partial y^2}(y,x) \geq 0$. (It can be verified that $v_t(s,x)$ and $w_t(s,x)$ are continuously differentiable by the Envelope Theorem. At any point where $v_t(s,x)$ or $w_t(s,x)$ is not twice differentiable, it can be checked that the left limit of the derivative at this point is less than or equal to the right limit. A similar approach can be used whenever we use second derivatives in the proofs.) Let $y_t^\circ(x)$ denote a minimizer of $w_t(y,x)$ over $y \in (-\infty, \infty)$. Then

$$v_{t+1}(s,x) = \begin{cases} w_{t+1}(s,x) - xs & \text{if } s \geq y_{t+1}^\circ(x), \\ w_{t+1}(y_{t+1}^\circ(x), x) - xs & \text{otherwise,} \end{cases}$$

and

$$\frac{\partial^2 v_{t+1}}{\partial s^2}(s,x) = \begin{cases} \frac{\partial^2 w_{t+1}}{\partial y^2}(s,x) & \text{if } s \geq y_{t+1}^\circ(x), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\frac{\partial^2 v_{t+1}}{\partial s^2}(s,x) \geq 0$ for all (s,x), and therefore

$$\frac{\partial^2 w_t}{\partial y^2}(y,x) = L''(y) + \beta \int_\xi \int_\epsilon \frac{\partial^2 v_{t+1}}{\partial s^2}(y - \xi, f(\epsilon)x + g(\epsilon))\Psi_t(d\epsilon)\phi(\xi)d\xi \geq 0.$$

By backward induction, we have $\frac{\partial^2 w_t}{\partial y^2}(y,x) \geq 0$ for all x and $t = 1, \ldots, T$. □

Proof of Proposition 2: We first prove by backward induction that $\frac{\partial^2 v_t}{\partial s \partial x}(s,x) \in [-1,1]$ for all (s,x) and $t = 1, \ldots, T+1$. It is true when $t = T+1$ because $v_{T+1}(s,x) = 0$. Suppose $\frac{\partial^2 v_{t+1}}{\partial s \partial x}(s,x) \in [-1,1]$ for all (s,x). By (2) we have $v_t(s,x) = w_t(y_t^*(s,x),x) - sx$. By Proposition 1 if $s \leq y_t^\circ(x)$, then $v_t(s,x) = w_t(y_t^\circ(x),x) - sx$ and $\frac{\partial^2 v_t}{\partial s \partial x}(s,x) = -1$. If $s > y_t^\circ(x)$, then

$$v_t(s,x) = w_t(s,x) - sx = L(s) + \beta \int_\xi \int_\epsilon v_{t+1}(s - \xi, f(\epsilon)x + g(\epsilon))\Psi_t(d\epsilon)\phi(\xi)d\xi,$$

and

$$\left|\frac{\partial^2 v_t}{\partial s \partial x}(s,x)\right| \leq \beta \int_\xi \int_\epsilon \left| f(\epsilon) \frac{\partial^2 v_{t+1}}{\partial s \partial x}(s-\xi, f(\epsilon)x + g(\epsilon)) \right| \Psi_t(d\epsilon)\phi(\xi)d\xi \leq E|f(\epsilon_t)| \leq 1.$$

The second inequality above follows from the inductive hypothesis.

Since $y_t^\circ(x)$ is a minimizer of $w_t(y,x)$ over $y \in (-\infty, \infty)$ and $w_t(y,x)$ is convex in y, it follows that $\frac{\partial w_t}{\partial y}(y_t^\circ(x), x) = 0$. Note that

$$\frac{\partial^2 w_t}{\partial y \partial x}(y,x) = 1 + \beta \int_\xi \int_\epsilon \frac{\partial^2 v_{t+1}}{\partial s \partial x}(y-\xi, f(\epsilon)x + g(\epsilon))\Psi_t(d\epsilon)\phi(\xi)d\xi \geq 1 - \beta \geq 0.$$

Hence, $\frac{\partial w_t}{\partial y}(y,x)$ is increasing in x. Therefore, for any $x' < x$, $\frac{\partial w_t}{\partial y}(y_t^\circ(x), x') \leq 0$. By the definition of $y_t^\circ(x')$ and the convexity of $w_t(y,x')$ in y, we have $y_t^\circ(x') \geq y_t^\circ(x)$. Thus, $y_t^\circ(x)$ is decreasing in x. □

Proof of Lemma 2: We have $v_{T+1}(s,x) = 0$, which is a concave function of x. Suppose $\frac{\partial^2 v_{t+1}}{\partial x^2}(s,x) \leq 0$ for all x. Using (3), we have

$$\frac{\partial^2 w_t}{\partial x^2}(y,x) = \beta \int_\xi \int_\epsilon f^2(\epsilon) \frac{\partial^2 v_{t+1}}{\partial x^2}(y-\xi, f(\epsilon)x + g(\epsilon))\Psi_t(d\epsilon)\phi(\xi)d\xi \leq 0.$$

Thus, $w_t(y,x)$ is a concave function of x. Since concavity is preserved under minimization, $v_t(s,x) = \min_{y \geq s}\{w_t(y,x)\} - xs$ is also a concave function of x. By backward induction, $\frac{\partial^2 v_t}{\partial x^2}(s,x) \leq 0$ for all x and $t = 1, \ldots, T+1$ and $v_t(s,x)$ is convex in x. □

Proof of Lemma 4: The proof is similar to that of Proposition 2. □

Proof of Lemma 5: We first show that $L(\mathbf{y})$ is a convex and submodular function of $\mathbf{y}$. We have

$$\frac{\partial L}{\partial y^i}(\mathbf{y}) = \begin{cases} h^i \int_0^{y^i} \phi(\xi)d\xi - (\sum_{j \neq i} h^j + b) \int_{y^i}^\infty \phi(\xi)d\xi & \text{if } y^i \leq \min\{y^1, \ldots, y^{i-1}, y^{i+1}, \ldots, y^n\}, \\ h^i & \text{if } y^i > \min\{y^1, \ldots, y^{i-1}, y^{i+1}, \ldots, y^n\}, \end{cases}$$

which is decreasing in y^j for $i \leq j$. Therefore, $L(\mathbf{y})$ is a submodular function of $\mathbf{y}$. Note that $L(\mathbf{y})$ can be written in the following form

$$L(\mathbf{y}) = \widehat{L}(\widehat{y}) + \sum_{i=1}^{n} h^i (y^i - \widehat{y}),$$

where $\widehat{L}(y) = \sum_{i=1}^{n} h^i \int_0^y (y-\xi)\phi(\xi)d\xi + \int_y^\infty b(\xi - y)\phi(\xi)d\xi$. Let $\mathbf{e}^i$ be the ith unit vector of dimension n. Since $h^i \int_0^{y^i} \phi(\xi)d\xi - (\sum_{j\neq i} h^j + b)\int_{y^i}^\infty \phi(\xi)d\xi \leq h^i$, we have $\frac{\partial L}{\partial y^i}(\mathbf{y}) \leq h^i$ for all $\mathbf{y}$. Therefore,

$$L(\mathbf{y} + \mathbf{e}^i) - L(\mathbf{y}) \leq h^i. \tag{10}$$

Given $\mathbf{y}$, define $\bar{\mathbf{y}}^i = (\max\{y^1, y^i\}, \ldots, \max\{y^n, y^i\})$ for $i = 1, \ldots, n$. Using (10) we have

$$\begin{aligned} L(\mathbf{y}) &\geq L(\bar{\mathbf{y}}^i) + \sum_{j: y^j \leq y^i} h^j (y^j - y^i) \\ &= \widehat{L}(y^i) + \sum_{j: y^j > y^i} h^j (y^j - y^i) + \sum_{j: y^j \leq y^i} h^j (y^j - y^i) \\ &= \widehat{L}(y^i) + \sum_{j=1}^{n} h^j (y^j - y^i). \end{aligned} \tag{11}$$

To show $L(\mathbf{y})$ is convex in $\mathbf{y}$, we need to show that $\lambda L(\mathbf{y}) + (1-\lambda)L(\widetilde{\mathbf{y}}) \geq L(\lambda\mathbf{y} + (1-\lambda)\widetilde{\mathbf{y}})$ for any $\mathbf{y}$, $\widetilde{\mathbf{y}}$, and $\lambda \in [0,1]$. Suppose $\lambda y^k + (1-\lambda)\widetilde{y}^k = \min_{i=1,\ldots,n} \{\lambda y^i + (1-\lambda)\widetilde{y}^i\}$. Then

$$\begin{aligned} \lambda L(\mathbf{y}) + (1-\lambda)L(\widetilde{\mathbf{y}}) &\geq \lambda[\widehat{L}(y^k) + \sum_{j=1}^{n} h^j (y^j - y^k)] + (1-\lambda)[\widehat{L}(\widetilde{y}^k) + \sum_{j=1}^{n} h^j (\widetilde{y}^j - \widetilde{y}^k)] \\ &= \lambda\widehat{L}(y^k) + (1-\lambda)\widehat{L}(\widetilde{y}^k) + \sum_{j=1}^{n} h^j [(\lambda y^j + (1-\lambda)\widetilde{y}^j) - (\lambda y^k + (1-\lambda)\widetilde{y}^k)] \\ &\geq \widehat{L}(\lambda y^k + (1-\lambda)\widetilde{y}^k) + \sum_{j=1}^{n} h^j [(\lambda y^j + (1-\lambda)\widetilde{y}^j) - (\lambda y^k + (1-\lambda)\widetilde{y}^k)] \\ &= L(\lambda\mathbf{y} + (1-\lambda)\widetilde{\mathbf{y}}), \end{aligned}$$

where the first inequality follows from (11) and the second from the convexity of $\widehat{L}(y)$.

Next we prove that $v_t(\mathbf{s}, \mathbf{x})$ is a convex and submodular function of $\mathbf{s}$ for $t = 1, \ldots, T+1$. We prove this by backward induction. We have that $v_{T+1}(\mathbf{s}, \mathbf{x}) = 0$ is convex and submodular in $\mathbf{s}$.

Suppose that $v_{t+1}(\mathbf{s},\mathbf{x})$ is convex and submodular in $\mathbf{s}$. Then

$$w_t(\mathbf{y},\mathbf{x}) = \mathbf{x}'\mathbf{y} + L(\mathbf{y}) + \beta \int_{\xi} Ev_{t+1}(\mathbf{y}-\xi, \Lambda(\boldsymbol{\epsilon}_t)\mathbf{x} + \mathbf{g}(\boldsymbol{\epsilon}_t))\phi(\xi)d\xi$$

is a convex and submodular function of $\mathbf{y}$ since $L(\mathbf{y})$ is convex and submodular in $\mathbf{y}$ and $Ev_{t+1}(\mathbf{y}-\xi, \Lambda(\boldsymbol{\epsilon}_t)\mathbf{x} + \mathbf{g}(\boldsymbol{\epsilon}_t))$ is convex and submodular in $\mathbf{y}$ by the inductive assumption. Submodularity is preserved under minimization on a lattice, so $v_t(\mathbf{s},\mathbf{x}) = \min_{\mathbf{y}\geq\mathbf{s}} w_t(\mathbf{y},\mathbf{x})$ is submodular in $\mathbf{s}$.

To show $v_t(\mathbf{s},\mathbf{x})$ is convex in $\mathbf{s}$, we need to show that $\lambda v_t(\mathbf{s},\mathbf{x}) + (1-\lambda)v_t(\widetilde{\mathbf{s}},\mathbf{x}) \geq v_t(\lambda\mathbf{s}+(1-\lambda)\widetilde{\mathbf{s}},\mathbf{x})$ for any $\mathbf{s}$, $\widetilde{\mathbf{s}}$, and $\lambda \in [0,1]$. Let $\mathbf{y}_t^*(\mathbf{x}) = \arg\min_{\mathbf{y}\geq\mathbf{s}} w_t(\mathbf{y},\mathbf{x})$ and $\widetilde{\mathbf{y}}_t^*(\mathbf{x}) = \arg\min_{\mathbf{y}\geq\widetilde{\mathbf{s}}} w_t(\mathbf{y},\mathbf{x})$. Then we have

$$\begin{aligned}
\lambda v_t(\mathbf{s},\mathbf{x}) + (1-\lambda)v_t(\widetilde{\mathbf{s}},\mathbf{x}) &= \lambda[w_t(\mathbf{y}_t^*(\mathbf{x}),\mathbf{x}) - \mathbf{x}'\mathbf{s}] + (1-\lambda)[w_t(\widetilde{\mathbf{y}}_t^*(\mathbf{x}),\mathbf{x}) - \mathbf{x}'\widetilde{\mathbf{s}}] \\
&\geq w_t(\lambda\mathbf{y}_t^*(\mathbf{x}) + (1-\lambda)\widetilde{\mathbf{y}}_t^*(\mathbf{x}),\mathbf{x}) - \mathbf{x}'[\lambda\mathbf{s} + (\mathbf{1}-\lambda)\widetilde{\mathbf{s}}] \\
&\geq \min_{\mathbf{y}\geq\lambda\mathbf{s}+(\mathbf{1}-\lambda)\widetilde{\mathbf{s}}} w_t(\mathbf{y},\mathbf{x}) - \mathbf{x}'(\lambda\mathbf{s} + (\mathbf{1}-\lambda)\widetilde{\mathbf{s}}) \\
&= v_t(\lambda\mathbf{s} + (1-\lambda)\widetilde{\mathbf{s}},\mathbf{x}).
\end{aligned}$$

The first inequality is due to the convexity of $w_t(\mathbf{y},\mathbf{x})$ in $\mathbf{y}$ and the second inequality is due to the fact that $\lambda\mathbf{y}_t^*(\mathbf{x}) + (1-\lambda)\widetilde{\mathbf{y}}_t^*(\mathbf{x}) \geq \lambda\mathbf{s} + (\mathbf{1}-\lambda)\widetilde{\mathbf{s}}$. □

Proof of Proposition 4: We only need to show that $w_t(\mathbf{y},\mathbf{x})$ is supermodular in (y^i, x^j) for $t = 1,\ldots,T$ and $i,j = 1,\ldots,n$. The proof of this is similar to that of Proposition 2. □

Proof of Theorem 4: We will show only that $v_t(\mathbf{s},\mathbf{x})$ is concave in $\mathbf{x}$ for $t = 1,\ldots,T+1$. The rest of the proof is similar to the proofs of Theorem 1, Corollary 1, and Lemma 3. We prove the concavity by induction.

For $t = T+1$, we have $v_{T+1}(\mathbf{s},\mathbf{x}) = 0$. Suppose $v_{t+1}(\mathbf{s},\mathbf{x})$ is concave in $\mathbf{x}$. Then

$$w_t(\mathbf{y},\mathbf{x}) = \mathbf{x}'\mathbf{y} + L(\mathbf{y}) + \beta \int_{\xi} Ev_{t+1}(\mathbf{y}-\xi, \Lambda(\boldsymbol{\epsilon}_t)\mathbf{x} + \mathbf{g}(\boldsymbol{\epsilon}_t))\phi(\xi)d\xi$$

is a concave function of $\mathbf{x}$. Concavity is preserved under minimization, so $v_t(\mathbf{s}, \mathbf{x})$ is concave in $\mathbf{x}$ for $t = 1, \ldots, T+1$. $\square$

Proof of Lemma 6: We prove this by induction. Suppose $\mathbf{X}_t \leq_{sm} \widehat{\mathbf{X}}_t$. Let $u(\cdot)$ be an arbitrary supermodular function and fix $\mathbf{x}$. Then for case (a), we have

$$Eu(\mathbf{X}_{t+1}(\mathbf{x})) = Eu(\Lambda(\boldsymbol{\epsilon}_t)\mathbf{x} + \mathbf{g}(\boldsymbol{\epsilon}_t)) = Eu(\Lambda(\widehat{\epsilon}_\mathbf{t})\mathbf{x} + \mathbf{g}(\widehat{\epsilon}_\mathbf{t})) = Eu(\widehat{\mathbf{X}}_{t+1}(\mathbf{x})).$$

For case (b), let $\tilde{u}(\boldsymbol{\epsilon}) = u(\Lambda(\boldsymbol{\epsilon})\mathbf{x} + \mathbf{g}(\boldsymbol{\epsilon}))$. Recall that each $f^i(\cdot)$ is assumed to be a constant, say a^i, and therefore $\Lambda(\boldsymbol{\epsilon}) = A$ where A is the matrix with a^i in the ith diagonal position for $i = 1, \ldots, n$ and zeros elsewhere. Hence $\tilde{u}(\boldsymbol{\epsilon}) = u(A\mathbf{x} + \mathbf{g}(\boldsymbol{\epsilon}))$.

We will now argue that $\tilde{u}(\boldsymbol{\epsilon})$ is supermodular. Suppose that $\varepsilon, \delta > 0$ and $i \neq j$. We have

$$\begin{aligned}
&\tilde{u}(\boldsymbol{\epsilon}+\varepsilon\mathbf{e}^i + \delta\mathbf{e}^j) - \tilde{u}(\boldsymbol{\epsilon} + \varepsilon\mathbf{e}^i) - \tilde{u}(\boldsymbol{\epsilon} + \delta\mathbf{e}^j) + \tilde{u}(\boldsymbol{\epsilon}) \\
&\quad = u(A\mathbf{x} + \mathbf{g}(\boldsymbol{\epsilon} + \varepsilon\mathbf{e}^i + \delta\mathbf{e}^j)) - u(A\mathbf{x} + \mathbf{g}(\boldsymbol{\epsilon} + \varepsilon\mathbf{e}^i)) - u(A\mathbf{x} + \mathbf{g}(\boldsymbol{\epsilon} + \delta\mathbf{e}^j)) + u(A\mathbf{x} + \mathbf{g}(\boldsymbol{\epsilon})) \\
&\quad = u(\mathbf{z} + \tilde{\varepsilon}\mathbf{e}^i + \tilde{\delta}\mathbf{e}^j) - u(\mathbf{z} + \tilde{\varepsilon}\mathbf{e}^i) - u(\mathbf{z} + \tilde{\delta}\mathbf{e}^j) + u(\mathbf{z}) \\
&\quad \geq 0
\end{aligned}$$

where we define $\mathbf{z} = A\mathbf{x} + \mathbf{g}(\boldsymbol{\epsilon})$, $\tilde{\varepsilon} = g^i(\epsilon^i + \varepsilon) - g^i(\epsilon^i)$, and $\tilde{\delta} = g^j(\epsilon^j + \delta) - g^j(\epsilon^j)$. The inequality above follows because $u(\cdot)$ is supermodular and because $\tilde{\epsilon}$ and $\tilde{\delta}$ have the same sign owing to the assumption that $g^i(\cdot)$ and $g^j(\cdot)$ are either both decreasing or both increasing. Hence we have established that $\tilde{u}(\boldsymbol{\epsilon})$ is supermodular.

As a consequence,

$$Eu(\mathbf{X}_{t+1}(\mathbf{x})) = E\tilde{u}(\boldsymbol{\epsilon}_t) \leq E\tilde{u}(\widehat{\epsilon}_\mathbf{t}) = Eu(\widehat{\mathbf{X}}_{t+1}(\mathbf{x})).$$

Thus, for both cases (a) and (b), we have $\mathbf{X}_{t+1}(\mathbf{x}) \leq_{sm} \widehat{\mathbf{X}}_{t+1}(\mathbf{x})$. Let $\eta(\mathbf{x}) = Eu(\mathbf{X}_{t+1}(\mathbf{x}))$ and $\widehat{\eta}(\mathbf{x}) = Eu(\widehat{\mathbf{X}}_{t+1}(\mathbf{x}))$. Then, we have $\eta(\mathbf{x}) \leq \widehat{\eta}(\mathbf{x})$ for all $\mathbf{x}$. In addition, it can be verified that $\widehat{\eta}(\mathbf{x})$

is a supermodular function of $\mathbf{x}$ (here we use $f^i(\epsilon^i)f^j(\epsilon^j) \geq 0$). Hence,

$$Eu(\mathbf{X}_{t+1}) = E\eta(\mathbf{X}_t) \leq E\widehat{\eta}(\mathbf{X}_t) \leq E\widehat{\eta}(\widehat{\mathbf{X}}_t) = Eu(\widehat{\mathbf{X}}_{\mathbf{t+1}}).$$

The second inequality holds because $\mathbf{X}_t \leq_{sm} \widehat{\mathbf{X}}_t$. This completes the induction and the proof. $\square$

Proof of Lemma 7: We prove the result by backward induction. We have that $v_{T+1}(\mathbf{s}, \mathbf{x}) = 0$ is trivially submodular in $\mathbf{x}$ for all $\mathbf{s}$. Suppose $v_{t+1}(\mathbf{s}, \mathbf{x})$ is submodular in $\mathbf{x}$ for all $\mathbf{s}$ and consider i, j with $i \neq j$. We will establish that $\frac{\partial^2 v_t}{\partial x^i \partial x^j}(\mathbf{s}, \mathbf{x}) \leq 0$.

In period t, we need to solve the optimization problem

$$\min \; w_t(\mathbf{y}, \mathbf{x}) = \mathbf{x}'\mathbf{y} + L(\mathbf{y}) + \beta \int_{\xi} E v_{t+1}(\mathbf{y} - \xi, \Lambda(\boldsymbol{\epsilon}_t)\mathbf{x} + \mathbf{g}(\boldsymbol{\epsilon}_t))\phi(\xi) d\xi$$

$$\text{s.t. } \; \mathbf{y} \geq \mathbf{s}.$$

The optimal solution $\mathbf{y}_t^*(\mathbf{s}, \mathbf{x}) = (y_t^{*1}(\mathbf{s}, \mathbf{x}), \ldots, y_t^{*n}(\mathbf{s}, \mathbf{x}))$ satisfies the KKT conditions:

$$\begin{aligned} \frac{\partial w_t}{\partial y^k}(\mathbf{y}, \mathbf{x}) - \lambda^k = 0 \quad & \text{for } k = 1, \ldots, n, \\ \lambda^k(y^k - s^k) = 0 \quad & \text{for } k = 1, \ldots, n, \\ \lambda^k \geq 0 \quad & \text{for } k = 1, \ldots, n. \end{aligned}$$

By Theorem 3, for each $k = 1, \ldots, n$, we know that $y_t^{*k}(\mathbf{s}, \mathbf{x}) = \max\{s^k, y_t^k(\mathbf{s}^{-k}, \mathbf{x})\}$, where $y_t^k(\mathbf{s}^{-k}, \mathbf{x})$ is the optimal base-stock level. If $s^k < y_t^k(\mathbf{s}^{-k}, \mathbf{x})$, then $y_t^{*k}(\mathbf{s}, \mathbf{x}) = y_t^k(\mathbf{s}^{-k}, \mathbf{x}) > s^k$. Thus, $\lambda^k = 0$ and $\frac{\partial w_t}{\partial y^k}(\mathbf{y}_t^*(\mathbf{s}, \mathbf{x}), \mathbf{x}) = 0$ by the KKT conditions. On the other hand, if $s^k \geq y_t^k(\mathbf{s}^{-k}, \mathbf{x})$, then $y_t^{*k}(\mathbf{s}, \mathbf{x}) = s^k$ and therefore $\frac{\partial y_t^{*k}}{\partial x^i}(\mathbf{s}, \mathbf{x}) = 0$ for any i.

Since $v_t(\mathbf{s}, \mathbf{x}) = w_t(\mathbf{y}_t^*(\mathbf{s}, \mathbf{x}), \mathbf{x}) - \mathbf{s}'\mathbf{x}$, we have

$$\begin{aligned} \frac{\partial v_t}{\partial x^i}(\mathbf{s}, \mathbf{x}) =& \sum_{k=1}^{n} \frac{\partial w_t}{\partial y^k}(\mathbf{y}_t^*(\mathbf{s}, \mathbf{x}), \mathbf{x}) \frac{\partial y_t^{*k}}{\partial x^i}(\mathbf{s}, \mathbf{x}) + \frac{\partial w_t}{\partial x^i}(\mathbf{y}_t^*(\mathbf{s}, \mathbf{x}), \mathbf{x}) - s^i \\ =& \frac{\partial w_t}{\partial x^i}(\mathbf{y}_t^*(\mathbf{s}, \mathbf{x}), \mathbf{x}) - s^i. \end{aligned}$$

Letting $I = \{k : s^k < y_t^k(\mathbf{s}^{-k}, \mathbf{x})\}$, we obtain

$$\frac{\partial^2 v_t}{\partial x^i \partial x^j}(\mathbf{s}, \mathbf{x}) = \sum_{k \in I} \frac{\partial^2 w_t}{\partial x^i \partial y^k}(\mathbf{y}_t^*(\mathbf{s}, \mathbf{x}), \mathbf{x}) \frac{\partial y_t^k}{\partial x^i}(\mathbf{s}^{-i}, \mathbf{x}) + \frac{\partial^2 w_t}{\partial x^i \partial x^j}(\mathbf{y}_t^*(\mathbf{s}, \mathbf{x}), \mathbf{x}). \qquad (12)$$

For the second term on the right side of (12), by the inductive hypothesis we have

$$\frac{\partial^2 w_t}{\partial x^i \partial x^j}(\mathbf{y}_t^*(\mathbf{s}, \mathbf{x}), \mathbf{x}) = \beta \int_{\xi} E\left[f^i(\epsilon_t^i) f^j(\epsilon_t^j) \frac{\partial^2 v_{t+1}}{\partial x_i \partial x_j}(\mathbf{y}_t^*(\mathbf{s}, \mathbf{x}) - \xi, \Lambda(\boldsymbol{\epsilon}_t)\mathbf{x} + \mathbf{g}(\boldsymbol{\epsilon}_t)) \right] \phi(\xi) d\xi \leq 0.$$

If we can establish that the first term on the right side of (12) is non-positive as well, then we will be done with the proof. To this end, note that $\frac{\partial w_t}{\partial y^k}(\mathbf{y}_t^*(\mathbf{s}, \mathbf{x}), \mathbf{x}) = 0$ for $k \in I$. Differentiating with respect to x^i yields

$$\sum_{\ell \in I} \frac{\partial^2 w_t}{\partial y^\ell \partial y^k}(\mathbf{y}_t^*(\mathbf{s}, \mathbf{x}), \mathbf{x}) \frac{\partial y_t^\ell}{\partial x^i}(\mathbf{s}^{-i}, \mathbf{x}) + \frac{\partial^2 w_t}{\partial x^i \partial y^k}(\mathbf{y}_t^*(\mathbf{s}, \mathbf{x}), \mathbf{x}) = 0.$$

Summing the preceding over $k \in I$ we get

$$\begin{aligned} \sum_{k \in I} \frac{\partial^2 w_t}{\partial x^i \partial y^k}(\mathbf{y}_t^*(\mathbf{s}, \mathbf{x}), \mathbf{x}) \frac{\partial y_t^k}{\partial x^i}(\mathbf{s}^{-i}, \mathbf{x}) &= -\sum_{k \in I} \sum_{\ell \in I} \frac{\partial^2 w_t}{\partial y^\ell \partial y^k}(\mathbf{y}_t^*(\mathbf{s}, \mathbf{x}), \mathbf{x}) \frac{\partial y_t^\ell}{\partial x^i}(\mathbf{s}^{-i}, \mathbf{x}) \frac{\partial y_t^k}{\partial x^i}(\mathbf{s}^{-i}, \mathbf{x}) \\ &= -\mathbf{z}' U \mathbf{z}, \end{aligned}$$

where $\mathbf{z}$ is the $|I|$-vector with entries $\frac{\partial y_t^\ell}{\partial x^i}(\mathbf{s}^{-i}, \mathbf{x})$ for $\ell \in I$, and U is the $|I| \times |I|$ matrix with entries $\frac{\partial^2 w_t}{\partial y^\ell \partial y^k}(\mathbf{y}_t^*(\mathbf{s}, \mathbf{x}), \mathbf{x})$ for $\ell, k \in I$. The matrix U is positive semidefinite because $w_t(\mathbf{y}, \mathbf{x})$ is convex in $\mathbf{y}$. As a consequence, $-\mathbf{z}' U \mathbf{z} \leq 0$. This establishes that the first term on the right side of (12) is non-positive and therefore completes the proof. $\square$