Inventory Pooling
Distributed versus Pooled Inventory

Versus

Assembly factory

Distribution center 1

Retailers

Assembly factory

Consolidated
distribution center

Retailers
Examples of Inventory Pooling

- Centralized warehousing (physical/virtual)
- Common components
- Delayed differentiation
- Product/component substitution
- E-tailing
The Impact of Inventory Pooling

- Impact on safety stocks
- Impact on cycle stocks
Impact on Safety Stocks

The distributed system:
A system with $N$ different inventory locations, each facing independent demand $X_i (i=1,\ldots, N)$, normally distributed with mean $\mu$ and standard deviation $\sigma$ and each with similar shortage and overage costs $c_s$ and $c_o$.  

$$Q_i^* = \mu + z_\alpha \sigma, \text{ for } i = 1,\ldots, N$$

$$Y_i(Q_i^*) = (c_s + c_o)\sigma \phi(z_\alpha), \text{ for } i = 1,\ldots, N$$

$$Y(Q_1^*,\ldots,Q_N^*) = N (c_s + c_o)\sigma \phi(z_\alpha)$$

Safety stock $= Nz_\alpha \sigma$
Impact on Safety Stocks  
(continued...)

The pooled system:
A system with a single inventory location from which all demand is satisfied,

\[
X = \sum_{i=1}^{N} X_i \\
Var(X) = Var(\sum_{i=1}^{N} X_i) = NVar(X_i) = N\sigma^2 \\
Q^* = N\mu + z_\alpha \sqrt{N}\sigma \\
Y(Q^*) = (c_s + c_o)\sqrt{N}\sigma \phi(z_\alpha) \\
\text{Safety stock} = \sqrt{N} z_\alpha \sigma
\]
Inventory pooling reduces both the optimal cost and the optimal amount of safety stock by a factor of $\sqrt{N}$. 

Impact on Safety Stocks (continued...)
Consider two locations with demand $X_1$ and $X_2$ with mean $\mu_1$ and $\mu_2$, standard deviation $\sigma_1$ and $\sigma_2$. Then

\[
Var(X) = Var(X_1 + X_2) = E \left[ (X_1 + X_2 - E(X_1 + X_2))^2 \right] \\
= Var(X_1) + Var(X_2) + 2E \left[ (X_1 - \mu_1)(X_2 - \mu_2) \right]
\]

Let

\[
Cov(X_1, X_2) = E \left[ [X_1 - \mu_1][X_2 - \mu_2] \right]
\]

Then

\[
Var(X) = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)
\]
What Happens if the Demands are Correlated? (Continued...)

- If \( \text{Cov}(X_1, X_2) > 0 \), \( X_1 \) and \( X_2 \) are positively correlated
- If \( \text{Cov}(X_1, X_2) < 0 \), \( X_1 \) and \( X_2 \) are negatively correlated
- If \( \text{Cov}(X_1, X_2) = 0 \), \( X_1 \) and \( X_2 \) are uncorrelated
What Happens if the Demands are Correlated?

Let

\[ \rho_{12} = \text{Cov}(\frac{X_1 - \mu_1}{\sigma_1}, \frac{X_2 - \mu_2}{\sigma_2}) \]

Then

\[ \rho_{12} = \frac{\text{Cov}(X_1, X_2)}{\sigma_1 \sigma_2} \]

\[ \text{Cov}(X_1, X_2) = \sigma_1 \sigma_2 \rho_{12} \]

\[ (-1 \leq \rho_{12} \leq 1) \]
What Happens if the Demands are Correlated?

If the demands $X_1$ and $X_2$ are pooled in a single location, then

$$Q^* = \mu_1 + \mu_2 + z_\alpha \sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1 \sigma_2 \rho_{12}}$$

$$Y(Q^*) = (c_o + c_s) \sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1 \sigma_2 \rho_{12}} \phi(z_\alpha)$$

If the demands $X_1$ and $X_2$ are satisfied from separate locations, then

$$Q_i^* = \mu_i + z_\alpha \sigma_i \quad \text{for } i = 1, 2$$

$$Y(Q_1^*, Q_2^*) = (c_s + c_o)(\sigma_1 + \sigma_2) \phi(z_\alpha)$$
If $\rho_{12} = -1$,

$$Q^* = \mu_1 + \mu_2 + z_\alpha \left| \sigma_1 - \sigma_2 \right|$$

$$Y(Q^*) = (c_o + c_s) \left| \sigma_1 - \sigma_2 \right| \phi(z_\alpha) < Y(Q^*_1, Q^*_1)$$

If $\rho_{12} = 1$,

$$Q^* = \mu_1 + \mu_2 + z_\alpha (\sigma_1 + \sigma_2)$$

$$Y(Q^*) = (c_o + c_s)(\sigma_1 + \sigma_2)\phi(z_\alpha) = Y(Q^*_1, Q^*_1)$$

If $\rho_{12} = 0$,

$$Q^* = \mu_1 + \mu_2 + z_\alpha \sqrt{\sigma_1^2 + \sigma_2^2}$$

$$Y(Q^*) = (c_o + c_s)\sqrt{\sigma_1^2 + \sigma_2^2} \phi(z_\alpha) < Y(Q^*_1, Q^*_1)$$
The Case of Identical demands

\((\mu_1 = \mu_2 = \mu; \sigma_1 = \sigma_2 = \sigma)\)

*If* \(\rho_{12} = -1,*

\(Q^* = 2\mu\)

\(Y(Q^*) = 0\)

*If* \(\rho_{12} = 1,*

\(Q^* = 2\mu + 2z_\alpha \sigma\)

\(Y(Q^*) = 2(c_o + c_s)\sigma\phi(z_\alpha)\)

*If* \(\rho_{12} = 0,*

\(Q^* = 2\mu + \sqrt{2}\sigma z_\alpha\)

\(Y(Q^*) = \sqrt{2}(c_o + c_s)\sigma\phi(z_\alpha)\)
The Case of Identical demands

\( \mu_1 = \mu_2 = \mu; \ \sigma_1 = \sigma_2 = \sigma \)

If \(-1 < \rho_{12} < 1\),

\[
Q^* = 2\mu + z_\alpha \sigma \sqrt{2(1 + \rho_{12})}
\]

\[
Y(Q^*) = (c_o + c_s)\sigma \sqrt{2(1 + \rho_{12})}\phi(z_\alpha)
\]
Inventory pooling is most beneficial when the demands have perfect negative correlation. On the other hand, there is no value to pooling when the demands have perfect positive correlation.
Impact on Cycle Stocks

The distributed system:
A system with $N$ different inventory locations, each facing a deterministic demand rate $D$ and each with ordering cost $A$ and holding cost $h$

$$Y(Q_1^*,...,Q_N^*) = N\sqrt{2ADh}$$

Average inventory = $N\sqrt{AD/2h}$
Impact on Cycle Stocks

The pooled system:
A system with a single location, facing a deterministic demand rate $ND$, ordering cost $A$ and holding cost $h$

$$Q^* = \sqrt{2ADN/h}$$

$$Y(Q^*) = \sqrt{2ANDh}$$

Average inventory = $\sqrt{ADN/2h}$
Example

\[ N = 2 \]
\[ \delta = \frac{Y(Q_1^*, ..., Q_N^*) - Y(Q^*)}{Y(Q_1^*, ..., Q_N^*)} = 1 - \frac{1}{\sqrt{2}} = 0.29 \]

Total cost is reduced by 29%.

\[ N = 4 \]
\[ \delta = 1 - \frac{1}{\sqrt{3}} = 0.42 \]

\[ N = 5 \]
\[ \delta = 1 - \frac{1}{\sqrt{4}} = 0.5 \]

\[ N = 10 \]
\[ \delta = 1 - \frac{1}{\sqrt{10}} = 0.68 \]
The Square Root Law

Average inventory increases proportionally to the square root of the number of locations in which inventory is held.
There is a diminishing effect to pooling. Most of the benefits of pooling occur by consolidating few locations. In most cases, total pooling may not be necessary.