IE 5531: Engineering Optimization I
Lecture 11: Midterm review

Prof. John Gunnar Carlsson

October 13, 2010
Administrivia

- Practice midterm 1 posted, solutions + a few more problems added later today
- PS3 solutions posted Friday
- PS1, PS2 handed back today
- Midterm 1 will be on 10/18/10
  - Lectures 1-9 covered
  - Open book, open notes
Lecture 2

- Equivalent forms of LP:

\[
\begin{align*}
\text{minimize} & \quad c^T x \quad s.t. \quad (1) \\
& \quad Ax \leq b \\
\text{minimize} & \quad c^T x \quad s.t. \quad (2) \\
& \quad Ax \leq b \\
& \quad x \geq 0 \\
\text{maximize} & \quad c^T x \quad s.t. \quad (3) \\
& \quad Ax = b \\
& \quad Dx \geq g \\
& \quad x \leq 0
\end{align*}
\]
Lecture 2

- **Standard form LP:**

  \[
  \text{minimize } c^T x \quad \text{s.t.} \\
  Ax = b \\
  x \geq 0
  \]

- **Tricks to change form:**
  - If \( x \) is free (no sign constraint), set \( x := x^+ - x^- \), with \( x^+, x^- \geq 0 \)
  - If \( a^T x \leq b \), set \( a^T x + s = b \), with \( s \geq 0 \)
  - If \( x \leq 3 \), set \( x + s = 3 \), with \( s \geq 0 \)
  - If \( x \geq 3 \), set \( x := x - 3 \)
Typical LP problems

- The *diet problem*: determine the cheapest possible diet that supplies all necessary nutrients
- Production planning: manufacture goods to make as large a profit as possible, without exceeding limits on available resources
Solving LPs graphically

Draw *half-spaces* corresponding to the constraints:
Solving LPs graphically

Draw half-spaces corresponding to the constraints:

\[ x_1 + 2x_2 \leq 3 \]

Points:
- \((0, 1.5)\)
- \((3, 0)\)
Solving LPs graphically

Draw *half-spaces* corresponding to the constraints:

\[ x_2 \]

(0,1.5) (1,1) (1.5,0)
Solving LPs graphically

Draw *half-spaces* corresponding to the constraints:
The graphical method

Draw level sets of the objective function (they’re lines orthogonal to $c$)
Facts about LP

- All LP problems fall into one of three classes:
  - Problem is *infeasible*: the feasible region is empty
  - Problem is *unbounded*: the feasible region is unbounded in the objective function direction
  - Problem is *feasible and bounded*:
    - There exists an *optimal solution* \( x^* \)
    - There may be a *unique* optimal solution or *multiple* optimal solutions
    - There is always at least one *corner* optimizer if the face has a corner
    - If a corner point is not worse than its neighboring corners, then it is optimal
Linearizing a problem

- Any piecewise-linear convex function can be minimized as an LP

\[
\begin{align*}
\text{minimize} \quad & z \\ 
\text{s.t.} \quad & z \geq c_i^T x + d_i \quad \forall i \in \{1, \ldots, m\} \\ 
& Ax \leq b
\end{align*}
\]

- Similarly, absolute values (and sums of absolute values) can be minimized

\[
\begin{align*}
\text{minimize} \quad & \sum_{i=1}^{n} c_i z_i \\ 
\text{s.t.} \quad & z_i \geq x_i \quad \forall i \in \{1, \ldots, n\} \\ 
& z_i \geq -x_i \quad \forall i \in \{1, \ldots, n\} \\ 
& Ax \leq b
\end{align*}
\]
The simplex method for a standard-form LP:

\[
\begin{align*}
\text{minimize} \quad & c^T x \\
\text{s.t.} \quad & Ax = b \\
& x \geq 0
\end{align*}
\]

Choose a set of \( m \) columns of \( A \), call it \( B \), and solve \( A_B x_B = b \), with \( x_N = 0 \)

- If \( x_B \geq 0 \), then the point \( x \) is a *corner point* or *basic feasible solution*
- We choose successively better basic sets \( B \) until we can’t make any more progress
Simplex tableau

\[ B \quad r = c - A^T (A_B^{-1})^T c_B \quad -c_B^T A_B^{-1} b \]

<table>
<thead>
<tr>
<th>basis indices</th>
<th>( A_B^{-1} A )</th>
<th>( A_B^{-1} b )</th>
</tr>
</thead>
</table>

The upper right element measures the current value of the objective function
Simplex algorithm

Initialize the simplex algorithm with a feasible basic set $B$, so that $x_B \geq 0$. Let $N$ be the remaining indices. Write the simplex tableau.

1. Test for termination. Find

$$r_e = \min_{j \in N} \{r_j\}$$

If $r_e \geq 0$, the solution is optimal. Otherwise, determine whether the column of $\bar{A}_e$ contains a positive entry. If not, the objective function is unbounded below. Otherwise, let $x_e$ be the entering basic variable.

2. Determine the outgoing variable. Use the MRT to determine the outgoing variable $x_o$.

3. Update the basic set. Update $B$ and $A_B$ and transform the problem to canonical form. Return to step 1.
Gauss-Jordan elimination

Given a simplex tableau, an outgoing variable $x_o$, and an entering variable $x_e$,

1. Divide all the entries in the row corresponding to $x_o$ by element $\bar{a}_{oe}$ (the “pivot element”), so that $\bar{a}_{oe} \mapsto 1$.

2. For all $i \neq o$, modify all other entries in the usual Gauss Jordan process:

   $$\bar{a}_{ij} \mapsto \bar{a}_{ij} - \frac{\bar{a}_{oj}}{\bar{a}_{oe}} \bar{a}_{ie}$$

3. Modify the right-hand side and the objective function row in the same way.

The above procedure allows us to find the optimal basic set without computing $A_B^{-1}$ at every step.
Example

Consider the problem (not in standard form)

$$\begin{align*}
\text{minimize} \quad & -3x_1 - 4x_2 \\
s.t. \quad & x_1 + x_2 \leq 4 \\
& 2x_1 + x_2 \leq 5 \\
& x_1, x_2 \geq 0
\end{align*}$$

We re-write this in standard form:

$$\begin{align*}
\text{minimize} \quad & -3x_1 - 4x_2 \\
s.t. \quad & x_1 + x_2 + x_3 = 4 \\
& 2x_1 + x_2 + x_4 = 5 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{align*}$$
Example

minimize \(-3x_1 - 4x_2\)

s.t. \(x_1 + x_2 + x_3 = 4\)
\(2x_1 + x_2 + x_4 = 5\)
\(x_1, x_2, x_3, x_4 \geq 0\)

- The first step is to choose a basic set \(B\), and then build \(\bar{A} = A_B^{-1}A\) and so forth in the tableau. What’s a good choice of \(B\)?
Example

\[
\begin{align*}
\text{minimize} & \quad -3x_1 - 4x_2 \\
\text{s.t.} & \quad x_1 + x_2 + x_3 = 4 \\
& \quad 2x_1 + x_2 + x_4 = 5 \\
& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

- The first step is to choose a basic set \( B \), and then build \( \bar{A} = A_B^{-1}A \) and so forth in the tableau. What’s a good choice of \( B \)?

- If we set \( B = \{3, 4\} \), then we find that \( A_B \) is just the identity matrix, so \( \bar{A} = A \)!

- If we are given a problem of the form \( \text{minimize } \mathbf{x} : A\mathbf{x} \leq \mathbf{b} \), where \( \mathbf{b} \geq \mathbf{0} \), this is always a good way to start
The canonical form shows us that this is not optimal. Pick \( x_1 \) as an entering basic variable. Using MRT, we see that \( \frac{4}{1} < \frac{5}{2} \), so \( x_4 \) will leave the basic set.

Next, we do Gauss-Jordan elimination:

1. Divide (III) by 2
2. Add 3\( \times \) (III) to (I)
3. Subtract (III) from (II)

### Tableau

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>−3</th>
<th>−4</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(II)</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>(III)</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>
The canonical form shows us that this is not optimal. Pick \( x_1 \) as an entering basic variable.

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>(I)</td>
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<td></td>
<td></td>
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<td>(II)</td>
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<tr>
<td>(III)</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

| (I) | 3 | 1 | 1 | 1 | 0 | 4 |
| (II) | 4 | 2 | 1 | 0 | 1 | 5 |
| (III) |   |   |   |   |   |   |
The canonical form shows us that this is not optimal. Pick $x_1$ as an entering basic variable

Using MRT, we see that $4/1 < 5/2$, so $x_4$ will leave the basic set
Tableau

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<tbody>
<tr>
<td>(I)</td>
<td>B</td>
<td>-3</td>
<td>-4</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(II)</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>(III)</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>

- The canonical form shows us that this is not optimal. Pick $x_1$ as an entering basic variable.
- Using MRT, we see that $4/1 < 5/2$, so $x_4$ will leave the basic set.
- Next, we do Gauss-Jordan elimination:
  1. Divide (III) by 2
  2. Add $3 \ast (III)$ to (I)
  3. Subtract (III) from (II)
The canonical form shows us that this is not optimal. Pick $x_2$ as an entering basic variable. Using MRT, we see that $rac{3}{2} < rac{5}{2}$, so $x_3$ will leave the basic set.

Next, we do Gauss-Jordan elimination:

Multiply (II) by 2.

Add $\frac{5}{2} \times (III)$ to (I).

Subtract $\frac{1}{2} \times (II)$ from (III).

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>(I)</td>
<td>B</td>
<td>0</td>
<td>$-5/2$</td>
<td>0</td>
<td>$3/2$</td>
</tr>
<tr>
<td>(II)</td>
<td>3</td>
<td>0</td>
<td>$1/2$</td>
<td>1</td>
<td>$-1/2$</td>
</tr>
<tr>
<td>(III)</td>
<td>1</td>
<td>1</td>
<td>$1/2$</td>
<td>0</td>
<td>$1/2$</td>
</tr>
</tbody>
</table>
The canonical form shows us that this is not optimal. Pick $x_2$ as an entering basic variable.
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Using MRT, we see that $\frac{3}{2} \cdot 1/2 < \frac{5}{2} \cdot 1/2$, so $x_3$ will leave the basic set.

<table>
<thead>
<tr>
<th>(I)</th>
<th>$B$</th>
<th>0</th>
<th>$-5/2$</th>
<th>0</th>
<th>$3/2$</th>
<th>$15/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(II)</td>
<td>3</td>
<td>0</td>
<td>$1/2$</td>
<td>1</td>
<td>$-1/2$</td>
<td>$3/2$</td>
</tr>
<tr>
<td>(III)</td>
<td>1</td>
<td>1</td>
<td>$1/2$</td>
<td>0</td>
<td>$1/2$</td>
<td>$5/2$</td>
</tr>
</tbody>
</table>
The canonical form shows us that this is not optimal. Pick \( x_2 \) as an entering basic variable.

Using MRT, we see that \( \frac{3/2}{1/2} < \frac{5/2}{1/2} \), so \( x_3 \) will leave the basic set.

Next, we do Gauss-Jordan elimination:

1. Multiply (II) by 2
2. Add \((5/2) \times (III)\) to (I)
3. Subtract \((1/2) \times (II)\) from (III)
The canonical form shows us that this is not optimal. Pick \( x_4 \) as an entering basic variable. Using MRT, we see that \( x_1 \) will leave the basic set (since \( 3 \div (-1) < 0 \)).

Next, we do Gauss-Jordan elimination:

1. Add \([III]\) to \([I]\)
2. Add \([III]\) to \([II]\)
The canonical form shows us that this is not optimal. Pick $x_4$ as an entering basic variable.

<table>
<thead>
<tr>
<th></th>
<th>$B$</th>
<th>0</th>
<th>0</th>
<th>5</th>
<th>$-1$</th>
<th>15</th>
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<tr>
<td>(I)</td>
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<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(II)</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>$-1$</td>
<td>3</td>
</tr>
<tr>
<td>(III)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$-1$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
The canonical form shows us that this is not optimal. Pick \( x_4 \) as an entering basic variable.

Using MRT, we see that \( x_1 \) will leave the basic set (since \( 3 / (-1) < 0 \)).

<table>
<thead>
<tr>
<th></th>
<th>( B )</th>
<th>0</th>
<th>0</th>
<th>5</th>
<th>-1</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I)</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>(II)</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>-1</td>
<td>3</td>
</tr>
<tr>
<td>(III)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
The canonical form shows us that this is not optimal. Pick $x_4$ as an entering basic variable.

Using MRT, we see that $x_1$ will leave the basic set (since $3/(-1) < 0$).

Next, we do Gauss-Jordan elimination:

1. Add (III) to (I)
2. Add (III) to (II)
The canonical form shows us that this is optimal, so we're done. The optimal basic set is $B = \{2, 4\}$ and that $x_2 = 4$ and $x_4 = 1$. The objective function value is 16.
The canonical form shows us that this is optimal, so we’re done.

The optimal basic set is $B = \{2, 4\}$ and that $x_2 = 4$ and $x_4 = 1$.

The objective function value is 16.
Lecture 6

- The *dual* of a standard-form LP

\[
\text{minimize } \mathbf{c}^T \mathbf{x} \quad \text{s.t.} \\
A \mathbf{x} = \mathbf{b} \\
\mathbf{x} \geq 0
\]

is

\[
\text{maximize } \mathbf{b}^T \mathbf{y} \quad \text{s.t.} \\
A^T \mathbf{y} \leq \mathbf{c}
\]

- The *weak duality theorem* says that \( \mathbf{c}^T \mathbf{x} \geq \mathbf{b}^T \mathbf{y} \) for all feasible \( \mathbf{x} \) and \( \mathbf{y} \)

- The *strong duality* says that if the primal and dual problem are both feasible, then \( \mathbf{c}^T \mathbf{x}^* = \mathbf{b}^T \mathbf{y}^* \)

- These statements hold for problems not in standard form also
<table>
<thead>
<tr>
<th>Primal</th>
<th>( \text{minimize} \ c^T x )</th>
<th>( \text{maximize} \ b^T y )</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( A )</td>
<td>( A^T )</td>
<td></td>
</tr>
<tr>
<td>constraints</td>
<td>( \geq b_i )</td>
<td>( \geq 0 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \leq b_i )</td>
<td>( \leq 0 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( = b_i )</td>
<td>( \text{free} )</td>
<td></td>
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<tr>
<td>variables</td>
<td>( \geq 0 )</td>
<td>( \leq c_j )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \leq 0 )</td>
<td>( \geq c_j )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \text{free} )</td>
<td>( = c_j )</td>
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</tr>
<tr>
<td>Dual</td>
<td>Finite optimum</td>
<td>Unbounded</td>
<td>Infeasible</td>
</tr>
<tr>
<td>---------------</td>
<td>----------------</td>
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<td>------------</td>
</tr>
<tr>
<td>Finite optimum</td>
<td>Possible</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Unbounded</td>
<td>X</td>
<td>X</td>
<td>Possible</td>
</tr>
<tr>
<td>Infeasible</td>
<td>X</td>
<td>Possible</td>
<td>Possible</td>
</tr>
</tbody>
</table>
# Duality examples

<table>
<thead>
<tr>
<th>Primal</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transportation problem</td>
<td>Pricing for goods</td>
</tr>
<tr>
<td>Maximum flow</td>
<td>Min cut</td>
</tr>
<tr>
<td>Production planning</td>
<td>Minimum acquisition</td>
</tr>
</tbody>
</table>

Also: multi-firm alliance, existence of core payment vector
Theorem

Let \( x \) and \( y \) be feasible solutions to the primal and dual problem, respectively. The vectors \( x \) and \( y \) are optimal solutions for the two respective problems if and only if

\[
\begin{align*}
    x_i > 0 & \implies A_i^T y = c_i \\
    A_i^T y < c_i & \implies x_i = 0
\end{align*}
\]

in other words,

\[
x_i \left( A_i^T y - c_i \right) = 0 \quad \forall i
\]
Complementary slackness

- Complementarity holds for LPs that are not in standard form as well.
- Let $a_j^T$ denote the $j$th row of $A$ and let $A_i$ denote the $i$th column of $A$.
- A primal-dual pair $x, y$ is optimal if and only if:

\[
y_j \left( a_j^T x - b_j \right) = 0 \quad \forall j \in \{1, \ldots, m\}
\]

\[
\left( c_i - y^T A_i \right) x_i = 0 \quad \forall i \in \{1, \ldots, n\}
\]

where $A \in \mathbb{R}^{m \times n}$

- In other words: take each variable and its associated constraint; one of the two must be 0.
If we change element $b_i$ of the RHS by an amount $\lambda$ (i.e. $b_i \mapsto b_i + \lambda$), the current basic set $B$ remains optimal so long as

$$x_B \geq -\lambda u$$

where $u$ is the $i$th column of $A_B^{-1}$

If we change element $c_j$ of the objective function by an amount $\lambda$, the current basic set remains optimal so long as

$$\begin{cases} r_N + \lambda \bar{e}_j \geq 0 & \text{if } j \in N \\ r_N - \lambda A_N^T (A_B^{-1})^T \bar{e}_j \geq 0 & \text{if } j \in B \end{cases}$$

where $\bar{e}_j = (e_j)_N$, $(e_j)_B$ respectively
Consider the linear program

\[
\text{maximize } x_1 + 2x_2 \\
\text{s.t. } x_1 \leq 1 \\
x_2 \leq 1 \\
x_1 + x_2 \leq 1.5 \\
x_1, x_2 \geq 0
\]
Example

The initial tableau is

<table>
<thead>
<tr>
<th></th>
<th>$B$</th>
<th>$-1$</th>
<th>$-2$</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
<td>0</td>
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<td>5</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1.5</td>
</tr>
</tbody>
</table>

The final tableau is

<table>
<thead>
<tr>
<th></th>
<th>$B$</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>1</th>
<th>2.5</th>
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<td>0</td>
<td>1</td>
<td>1</td>
<td>−1</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>−1</td>
<td>1</td>
<td>0.5</td>
<td></td>
</tr>
</tbody>
</table>
Example

- The optimal basis in the preceding example was \( \{1, 2, 3\} \) with \( x_B = (0.5, 1, 0.5) \); we therefore have

\[
A_B^{-1} = \begin{pmatrix}
0 & -1 & 1 \\
0 & 1 & 0 \\
1 & 1 & -1
\end{pmatrix}; \quad A_N = \begin{pmatrix}
0 & 0 \\
1 & 0 \\
0 & 1
\end{pmatrix}; \quad r_N = \begin{pmatrix}
1 \\
1
\end{pmatrix}
\]

- The initial objective coefficients were \((-1, -2, 0, 0, 0)\); we'll change the first element, \( c_1 = -1 \), to \(-1 + \lambda \)

- Since \( x_1 \) is a basic variable, we require that

\[
\begin{pmatrix}
1 \\
1
\end{pmatrix} - \lambda \begin{pmatrix}
1 \\
1
\end{pmatrix} \geq 0
\]

and therefore \(-1 \leq \lambda \leq 1\)