Before you begin: This exam has 9 pages and a total of 5 problems. Make sure that all pages are present. To obtain credit for a problem, you must show all your work. If you use a formula to answer a problem, write the formula down. Do not open this exam until instructed to do so.
Problem 1: True or False (10 points)
State whether each of the following statements is true or false. Assume all auxiliary information given is correct. For each part, only your answer, which should be one of “True” or “False”, will be graded. Any explanation will not be read.

1. Although its worst-case running time is faster than that of the simplex method, the ellipsoid method is rarely used because it performs poorly in practical situations. True.

2. One disadvantage of the Wolfe line-search conditions is that, in any given iteration, we may erroneously remove local minimizers from the search space. False. (local minimizers have $\phi' (\alpha) = 0$, which is never discarded)

3. At a point satisfying the KKT conditions, if a Lagrange multiplier $\lambda$ of a constraint is 0, then that constraint must be inactive. False. Minimizing $x^2$ subject to $x \geq 0$ has a zero Lagrange multiplier but the constraint is active at optimality.

4. Consider the problem

\[
\begin{align*}
\text{minimize} & \quad f (x) \\
\text{s.t.} & \quad a^T x \leq b
\end{align*}
\]

where $f(x)$ is a convex function. If $x^*$ satisfies $a^T x^* < b$ and $\nabla f(x^*) = 0$, then $x^*$ must be a globally optimal solution to our problem. True.

5. Newton’s method, applied to the function below, should converge to the marked root of the function if our initial guess lies in the shaded interval. True.

![Graph showing the convergence of Newton's method to the marked root.](image)
Problem 2: Line search conditions (20 points) Suppose that you are performing a line search to find an unconstrained minimum of the function $f(x)$ in $\mathbb{R}^n$. At the current iteration, you have chosen your descent direction $d_k$ and you are currently trying to determine the optimal step length $\alpha_k$. Sketch the ranges of feasible step lengths in the figure below for the Goldstein conditions. You may assume that $\phi'(0) = \nabla f(x_k)^T d_k = -1$, as suggested by the figure. Assume that $c = 1/4$ and consequently that $1 - c = 3/4$. 
Problem 3: Cobb-Douglas Utility Function (20 points) The Cobb-Douglas production function is widely used in economics to represent the relationship between inputs and outputs of a firm. It takes the form $Y = AL^\alpha K^\beta$ where $Y$ represents output, $L$ labor, and $K$ capital. The parameters $\alpha$ and $\beta$ are constants that determine how production is scaled. The Cobb-Douglas function can also be applied to utility maximization and takes the general form $\prod_{i=1}^{N} x_i^{\alpha_i}$.

Consider the following utility maximization problem:

$$\text{maximize } u(x) = x_1^\alpha x_2^{1-\alpha} \quad \text{s.t.}$$
$$p_1 x_1 + p_2 x_2 \leq w$$
$$x_1, x_2 \geq 0$$

Here $p$ is a price vector for $x_1$ and $x_2$ and $w$ represents a user’s budget. Assume that $p > 0$ and $w > 0$.

1. Perform an appropriate transformation to turn this into a minimization problem with a convex objective function.

**Solution** Taking logarithms and multiplying by $-1$ we obtain the equivalent convex problem

$$\text{minimize } - [\alpha \log x_1 + (1 - \alpha) \log x_2] \quad \text{s.t.}$$
$$p_1 x_1 + p_2 x_2 \leq w$$
$$x_1, x_2 \geq 0$$

2. Write the KKT conditions and find an explicit solution for $x$ as a function of $\alpha$, $p$, and $w$. (Hint: can you assume that certain constraints must be tight? Can you assume that certain constraints must be slack?) Are these conditions sufficient for optimality?

**Solution** The KKT conditions for the transformed problem are

$$-\left( \frac{\alpha}{x_1} \right) + \lambda_0 \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 0$$
$$\lambda_0 (p_1 x_1 + p_2 x_2 - w) = 0$$
$$\lambda_1 x_1 = 0$$
$$\lambda_2 x_2 = 0$$
$$p_1 x_1 + p_2 x_2 \leq w$$
$$x_1, x_2 \geq 0$$

The constraint $p_1 x_1 + p_2 x_2 \leq w$ must be tight because the objective function of the transformed problem is monotonically decreasing in $x_1$ and $x_2$. The sign constraints on $x_1$ and $x_2$ are slack because otherwise the objective function of the transformed problem is $\infty$, and thus $\lambda_1 = \lambda_2 = 0$. The conditions above therefore tell us that

$$-\frac{\alpha}{x_1} + \lambda_0 p_1 = 0$$
$$-(1 - \alpha)/x_2 + \lambda_0 p_2 = 0$$
$$p_1 x_1 + p_2 x_2 = w$$

which admits the unique solution $\lambda_0^* = 1/w$ so that $x_1^* = \alpha w/p_1$ and $x_2^* = (1 - \alpha)w/p_2$.

3. Find the corresponding Lagrange multiplier for the budget constraint and give an interpretation of it.

**Solution** The Lagrange multiplier is $\lambda_0^* = 1/w$; the economic interpretation is that it represents the marginal benefit of one extra unit of budget.
Problem 4: A linearly constrained problem (25 points) Consider the following linearly constrained convex optimization problem:

\[
\begin{align*}
\text{minimize} & \quad x_1^2 - 6x_1 + x_2^3 - 3x_2 \\
\text{s.t.} & \quad x_1 + x_2 \leq 1 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

1. Derive the KKT conditions for this problem.

**Solution** The KKT conditions are

\[
\begin{align*}
2x_1 - 6 + \lambda_0 - \lambda_1 &= 0 \\
3x_2^2 - 3 + \lambda_0 - \lambda_2 &= 0 \\
\lambda_0 (x_1 + x_2 - 1) &= 0 \\
\lambda_1 x_1 &= 0 \\
\lambda_2 x_2 &= 0 \\
x_1 + x_2 &\leq 1 \\
x_1, x_2 &\geq 0
\end{align*}
\]

2. Determine whether or not \( x = (1/2, 1/2) \) is an optimal solution.

**Solution** If \( x = (1/2, 1/2) \), then by complementary slackness we must have \( \lambda_1 = \lambda_2 = 0 \). However, the first two equations tell us that

\[
\begin{align*}
2x_1 - 6 + \lambda_0 - \lambda_1 &= 0 \\
3x_2^2 - 3 + \lambda_0 - \lambda_2 &= 0
\end{align*}
\]

so that \( 2(1/2) - 6 + \lambda_0 = 3(1/2)^2 - 3 + \lambda_0 = 0 \), which cannot happen, and thus \( (1/2, 1/2) \) is not optimal.

3. Given that the optimal solution lies on a corner point of the feasible region, find the optimal solution to the above problem.

**Solution** The feasible region has three corner points: \( (0, 0) \), \( (1, 0) \), and \( (0, 1) \). By inspection we find that the optimal point is \( (1, 0) \).
Problem 5: Interior point method for LP (25 points)  
Suppose we want to solve the following linear program:

\[
\begin{align*}
\text{minimize} & \quad x_1 + x_2 & s.t. \\
& x_1 + x_2 + x_3 = 1 \\
& x_1, x_2, x_3 \geq 0
\end{align*}
\]

We will use a barrier method on the inequality constraints (not the primal-dual potential function!) to solve this problem.

1. Write the KKT conditions for the barrier-function relaxation to this linear program.

Solution  
The problem is

\[
\begin{align*}
\text{minimize} & \quad x_1 + x_2 - \mu (\log x_1 + \log x_2 + \log x_3) & s.t. \\
& x_1 + x_2 + x_3 = 1
\end{align*}
\]

and the KKT conditions are

\[
\begin{align*}
1 - \mu/x_1 + \lambda_0 &= 0 \\
1 - \mu/x_2 + \lambda_0 &= 0 \\
-\mu/x_3 + \lambda_0 &= 0 \\
x_1 + x_2 + x_3 &= 1
\end{align*}
\]

2. Show that for every \( \mu > 0 \), the optimal solution to the barrier-function relaxation is:

\[
\begin{align*}
x_1(\mu) &= \frac{1 + 3\mu - \sqrt{1 + 9\mu^2 - 2\mu}}{4} \\
x_2(\mu) &= x_1(\mu) \\
x_3(\mu) &= 1 - 2x_1(\mu)
\end{align*}
\]

Solution  
The KKT conditions give us four equations in four unknowns. Isolating \( \lambda_0 \) we can write

\[
\begin{align*}
1 - \mu/x_1 &= 1 - \mu/x_2 = -\mu/x_3 \\
x_1 + x_2 + x_3 &= 1
\end{align*}
\]

which tells us that \( x_2 = x_1 \) at optimality. Since \( x_1 + x_2 + x_3 = 1 \) this tells us that \( x_3 = 1 - 2x_1 \). To solve for \( x_1 \) we write

\[
1 - \mu/x_1 = -\mu/x_3 = -\mu/(1 - 2x_1)
\]

which gives a quadratic equation in \( x_1 \). Solving this equation gives us

\[
x_1 = \frac{1 + 3\mu \pm \sqrt{1 + 9\mu^2 - 2\mu}}{4}.
\]

We have two options for the \( \pm \) sign, since both give feasible solutions. However, the objective function at \( x_1 = x_2 = \frac{1 + 3\mu + \sqrt{1 + 9\mu^2 - 2\mu}}{4} \) is strictly greater than the objective function at \( x_1 = x_2 = \frac{1 + 3\mu - \sqrt{1 + 9\mu^2 - 2\mu}}{4} \), and the desired result follows.

3. Show that as \( \mu \to 0 \), the optimal solution to the barrier problem approaches the unique optimal solution.

Solution  
It is straightforward to verify that \( \lim_{\mu \to 0} (x_1(\mu), x_2(\mu), x_3(\mu)) = (0, 0, 1) \).

4. (Bonus; 5 points) Suppose that our objective function is just to minimize \( x_1 \). Where does the solution converge to now?

Solution  
Using the same approach the solution converges to \( (0, 1/2, 1/2) \).