# 6. Comparing Alternatives

One of the main purposes of this course is to discuss how to make decisions in engineering economy.

Let us first consider a single period case.

Suppose that there are two projects: A and B. Project A requires an investment of HK$10K and will return in one year time HK$12K. The other project requires HK$100K investment and returns HK$115K next year.

Clearly, $\text{irr}_A = 20\%$ and $\text{irr}_B = 15\%$.

Suppose that your MARR is 10%.

What do you do?
First of all, we want to invest.

Why?

Because:

\[ PW_A(MARR) = -10,000 + 12,000/1.1 \approx 909 \]

and

\[ PW_B(MARR) = -100,000 + 115,000/1.1 \approx 4546 \]

Additionally, assume that:

- The projects are mutually exclusive.
- There is no limit on the budget.
According to their present value, the preference is clearly $B \succ A$.

According to the IRR, $A$ looks more attractive than $B$.

*Which consideration makes more sense?*

Let us start from the basis that:

Any investment is acceptable only when its PW under the MARR is positive.
So, we must justify the use of the additional $90,000 in Project $B$. In fact $B \setminus A$ requires an initial investment of $90,000$ and returns with $103,000$. Under the MARR its present worth is

$$-90,000 + \frac{103,000}{1.1} \approx 3,636$$

**Conclusion:** The use of additional $90,000$ in Project $B$ is justified, and so $B \succ A$.

**Careful:** The IRR can give some misleading conclusions!
To make our logic more explicit, consider the case where $A$ will return with $20K$ in one year.

Then, the incremental investment in $B$, as compared against $A$, is:
input $(100-10)K$; output $(115-20)K$.

Is this a good deal? (bearing in mind that we have the MARR!)

Well, the PW of the incremental investment, $B \setminus A$, under the MARR 10%, is

$$-(100 - 10)K + \frac{115 - 20}{1.1}K = -3.6K.$$ 

So, in this case, we should not do $B$! Rather, we should take $A$. 
Following this idea, we propose a general procedure for selecting one project among mutually exclusive alternatives (MEA), provided that the MARR is given.

The Incremental Investment Analysis:

- Order the projects according to their initial capital requirement.
- Let $\Delta$ be the difference of the first two remaining projects:
  - Delete the first project if
    \[
    IRR(\Delta) \geq MARR
    \]
  - Delete the second project if
    \[
    IRR(\Delta) < MARR
    \]
- Repeat until there is only one project left.
Computationally speaking, the above procedure maybe cumbersome. Also, if the IRR is non-unique, then it may be difficult to implement. The problem is lessened, if we observe that for $\Delta = B \setminus A$,

$$IRR(\Delta) \geq MARR$$

only if

$$PW_{B \setminus A}(MARR) = PW_B(MARR) - PW_A(MARR) \geq 0.$$  

This yields a simplified procedure based on the PW of the alternatives.
Projects Evaluation based on the PWs:

- Calculate the PWs under the MARR.
- Order the projects according to their initial capital requirement.
- Compute the PW of the difference of the first two remaining projects:
  - Delete the first project if the PW is positive;
  - Delete the second project if the PW is negative.
- Repeat until there is only one project left.

In fact, one will select the one with the highest (positive) PW value.
Example: Consider projects $A$, $B$ and $C$, all last for 10 years. The MARR is 10%.

<table>
<thead>
<tr>
<th></th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>initial</td>
<td>-$390K</td>
<td>-$920K</td>
<td>-$660K</td>
</tr>
<tr>
<td>annual</td>
<td>$69K</td>
<td>$167K</td>
<td>$133.5K</td>
</tr>
</tbody>
</table>

The order in terms of initial capital investments is: $A$, $C$ and $B$.

Let us solve the problem by the incremental IRR method.

We first compare $A$ and $C$ by considering $C \setminus A$:

<table>
<thead>
<tr>
<th>Project</th>
<th>Initial</th>
<th>Annual</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C \setminus A$</td>
<td>-$270K</td>
<td>$64.5K$</td>
</tr>
</tbody>
</table>

The IRR of this cash flow is $20.05\% > MARR$. So $C \succ A$. 
We delete $A$ from further consideration, and proceed to compare $C$ with $B$.

\[
\begin{array}{|c|c|c|}
\hline
\text{Project} & \text{Initial} & \text{Annual} \\
\hline
B \backslash C & -$260K & $33.5K \\
\hline
\end{array}
\]

The calculation shows that

\[
IRR_{B\backslash C} = 3.07\% < MARR.
\]

Hence we reject $B$, and $C$ is selected as the result.
Now we see what happens if we apply the PW method.

First of all, the PWs are computed:

\[
P W_A(10\%) = -$390K + $69K(P/A, 10\%, 10) \\
\quad \approx $34K
\]

\[
P W_B(10\%) = -$920K + $167K(P/A, 10\%, 10) \\
\quad \approx $106K
\]

\[
P W_C(10\%) = -$660K + $133.5K(P/A, 10\%, 10) \\
\quad \approx $160K
\]

So, they all qualify for further consideration.
In the increasing order of initial investment: $A$, $C$ and $B$.

We consider $C \setminus A$:

\[ PW_{C \setminus A} \approx 126K > 0 \]

Therefore we take $C$ and delete $A$.

Now consider $B \setminus C$:

\[ PW_{B \setminus C} \approx -54K < 0 \]

So, the additional budget needed for $B$ is worse than the MARR.

**Conclusion**: $C$ is most attractive under the MARR. In fact,

\[ C \succ B \succ A. \]
A mathematical question: *Can the ordering produced by the incremental method be inconsistent?*
That is, can it be that $A \succ B$ and $B \succ C$, but $C \succ A$?
The answer is: NO! *Why?* Because:

$$A \succ B \text{ iff } PW_A(MARR) > PW_B(MARR).$$

However, the preference relation is largely dependent on the MARR!
Consider again our first example. Suppose now that the MARR is 14.8%. Then,

$$PW_A(14.8\%) = -10 + 12/1.148 \approx 0.453K$$

and

$$PW_B(14.8\%) = -100 + 115/1.148 \approx 0.1742K$$

Hence, $A$ becomes preferable!
Note: Of course, the evaluation based on PW is equivalent to that based on FW or AW.

As we have seen in the first example, it is wrong to select the project with the highest IRR.

Example: Consider 6 projects \(A, B, C, D, E\) and \(F\). The duration for all projects are for 10 years and their cash flows are given as follows:

<table>
<thead>
<tr>
<th>Project</th>
<th>Initial Investment</th>
<th>Annual Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>-$900</td>
<td>$150</td>
</tr>
<tr>
<td>(B)</td>
<td>-$1,500</td>
<td>$276</td>
</tr>
<tr>
<td>(C)</td>
<td>-$2,500</td>
<td>$400</td>
</tr>
<tr>
<td>(D)</td>
<td>-$4,000</td>
<td>$925</td>
</tr>
<tr>
<td>(E)</td>
<td>-$5,000</td>
<td>$1,125</td>
</tr>
<tr>
<td>(F)</td>
<td>-$7,000</td>
<td>$1,425</td>
</tr>
</tbody>
</table>

Suppose that \(MARR = 10\%\).
First of all, we can calculate the IRR of these projects:

<table>
<thead>
<tr>
<th>Project</th>
<th>IRR</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>10.6%</td>
</tr>
<tr>
<td>B</td>
<td>13.0%</td>
</tr>
<tr>
<td>C</td>
<td>9.6%</td>
</tr>
<tr>
<td>D</td>
<td>19.1%</td>
</tr>
<tr>
<td>E</td>
<td>18.3%</td>
</tr>
<tr>
<td>F</td>
<td>15.6%</td>
</tr>
</tbody>
</table>

According to the IRR, Project $D$ turns out to be the best.

But how can we interpret this fact?

It is true that if the MARR is 19%, then $D$ is the only acceptable choice. But now, the MARR is much lower, so in fact we may actually waste some investment opportunity by investing in $D$. 
We go on carrying out the ‘incremental analysis’ which results in an ordering in PWs under the current MARR.

<table>
<thead>
<tr>
<th>Project</th>
<th>PW under MARR</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>21.69</td>
</tr>
<tr>
<td>B</td>
<td>196</td>
</tr>
<tr>
<td>C</td>
<td>-42.16</td>
</tr>
<tr>
<td>D</td>
<td>1683.8</td>
</tr>
<tr>
<td>E</td>
<td>6412.7</td>
</tr>
<tr>
<td>F</td>
<td>1756.1</td>
</tr>
</tbody>
</table>

Therefore, if we can only invest in one project, then it is $E$. The order is:

$$E \succ F \succ D \succ B \succ A \succ C$$

In fact $C$ is not qualified at all.
The matter will be quite different if we have a limited budget and/or the projects are not mutually exclusive.

For example, suppose that we have precisely $4,000 to invest. Then, one should certainly invest in Project $D$.

If the projects are not mutually exclusive, then the preference would be:

$$D \succ E \succ F \succ B \succ A$$

If IRRs are not unique, then one can use the ERR instead.
Comparing Alternatives (continue).

So far, we have made the following assumptions in our analysis:

- The alternatives are mutually exclusive
- The life-span of the alternatives are the same

The basic idea of the method is simple: We try to see whether any additional cent on a more expensive project is well spent, in comparison with the MARR.

This is termed the *incremental analysis*. 
In some applications indeed only one alternative is to be chosen.

*Example:* Suppose that one must install an air compressor among 4 choices for 5 years. Their respective expenses are as follows.

<table>
<thead>
<tr>
<th>Project</th>
<th>Price</th>
<th>Annual Cost</th>
<th>Resale Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1$</td>
<td>-100K</td>
<td>-29K</td>
<td>10K</td>
</tr>
<tr>
<td>$D_2$</td>
<td>-140K</td>
<td>-16.9K</td>
<td>14K</td>
</tr>
<tr>
<td>$D_3$</td>
<td>-148.2K</td>
<td>-14.8K</td>
<td>25.6K</td>
</tr>
<tr>
<td>$D_4$</td>
<td>-122K</td>
<td>-22.1K</td>
<td>14K</td>
</tr>
</tbody>
</table>

Suppose that MARR=20%.
First of all, this is a cost alternatives, meaning that we need only to see whether the benefit from any additional cost is more attractive than the \( MARR \) or not.

From \( D_1 \) to \( D_4 \), the cash flow is

<table>
<thead>
<tr>
<th>Price Diff</th>
<th>Annual Saving</th>
<th>End Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>-22K</td>
<td>6.9K</td>
<td>4K</td>
</tr>
</tbody>
</table>

Hence,

\[
IRR(\Delta _{D_4\setminus D_1}) = 20.5\%.
\]

The investment is justified.
Next we consider the incremental from $D_4$ to $D_2$.

<table>
<thead>
<tr>
<th>Price Diff</th>
<th>Annual Saving</th>
<th>End Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>-18K</td>
<td>5.2K</td>
<td>0K</td>
</tr>
</tbody>
</table>

$$IRR(\Delta_{D_2\backslash D_4}) = 12.3\%.$$  

The additional investment is NOT justified.
From $D_4$ to $D_3$.

<table>
<thead>
<tr>
<th>Price Diff</th>
<th>Annual Saving</th>
<th>End Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>-26.2K</td>
<td>7.3K</td>
<td>11.6K</td>
</tr>
</tbody>
</table>

$$IRR(\Delta_{D_3\setminus D_4}) = 20.4\%.$$  

The additional investment is justified.

Conclusion: $D_3$ is most attractive, provided that $MARR = 20\%$. 
We may as well use the PWs in the analysis.

The PWs under the MARR are:

<table>
<thead>
<tr>
<th>Project</th>
<th>PW(MARR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1$</td>
<td>-182.7084K</td>
</tr>
<tr>
<td>$D_2$</td>
<td>-185.5148K</td>
</tr>
<tr>
<td>$D_3$</td>
<td>-182.1722K</td>
</tr>
<tr>
<td>$D_4$</td>
<td>-182.4657K</td>
</tr>
</tbody>
</table>

Therefore, the least costly alternative is $D_3$.

It leads to the same conclusion.
*Example:* A downtown parking center was out of capacity. Catherine Jones, an ambitious new employee of an architectural engineering firm, was called to perform the project evaluation. Alternatives are:

- **P:** Keep and improve existing parking lot
- **B_1:** Construct one-story building
- **B_2:** Construct two-story building
- **B_3:** Construct three-story building

Expenses versus incomes:

<table>
<thead>
<tr>
<th>Project</th>
<th>Investment</th>
<th>Net Annual Income</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>-$200K</td>
<td>$22K</td>
</tr>
<tr>
<td>$B_1$</td>
<td>-$4M</td>
<td>$600K</td>
</tr>
<tr>
<td>$B_2$</td>
<td>-$5.55M</td>
<td>$720K</td>
</tr>
<tr>
<td>$B_3$</td>
<td>-$7.5M</td>
<td>$960K</td>
</tr>
</tbody>
</table>

Shuzhong Zhang
In 15 years time, the residual value of each alternative is first estimated as the same as its construction cost today.

Suppose that the MARR of the firm is 10%.

The IRR of each alternative is shown to be:

<table>
<thead>
<tr>
<th>Project</th>
<th>IRR</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>11%</td>
</tr>
<tr>
<td>$B_1$</td>
<td>15%</td>
</tr>
<tr>
<td>$B_2$</td>
<td>13%</td>
</tr>
<tr>
<td>$B_3$</td>
<td>12.8%</td>
</tr>
</tbody>
</table>

The management of the firm was tempted to choose $B_1$, which has the highest IRR.

Catherine discovered however that the alternatives are mutually exclusive. Therefore an incremental cost analysis is more appropriate.
This boils down to calculating the PWs under the MARR:

<table>
<thead>
<tr>
<th>Project</th>
<th>PWs under MARR</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>$15,214</td>
</tr>
<tr>
<td>$B_1$</td>
<td>$1,521,260</td>
</tr>
<tr>
<td>$B_2$</td>
<td>$1,255,062</td>
</tr>
<tr>
<td>$B_3$</td>
<td>$1,597,356</td>
</tr>
</tbody>
</table>

Therefore, Catherine recommended $B_3$. Moreover, she remarked that if the estimation of the residual values were wrong, say they were half of the construction costs, then:
### Project PWs under MARR IRR

<table>
<thead>
<tr>
<th>Project</th>
<th>PWs under MARR</th>
<th>IRR</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>-$8,726</td>
<td>9.3%</td>
</tr>
<tr>
<td>$B_1$</td>
<td>$1,042,460</td>
<td>13.8%</td>
</tr>
<tr>
<td>$B_2$</td>
<td>$590,727</td>
<td>11.6%</td>
</tr>
<tr>
<td>$B_3$</td>
<td>$699,606</td>
<td>11.4%</td>
</tr>
</tbody>
</table>

In that case, she would recommend $B_1$.

The management sees her merits and she was quickly promoted upwards in the organization.
What to do when the duration (‘useful life’) of the projects are different?

Obviously it is no longer “fair” only to compare the PWs. For example, one project, $P_1$, has a duration of 2 years, and the other project, $P_2$, has a duration of 4 years.

It can be that $PW_{P_1}(MARR) < PW_{P_2}(MARR)$.

But, if we adopt $P_1$ and replace the last two years with $MARR$, then the combined project is more attractive than $P_2$. 
We first consider the situation when a project can be repeated.
So we can try to fill up the whole time horizon with the same repeated projects.
Let the duration of $P_1$ be $N_1$, and the duration of $P_2$ be $N_2$.
Let the least common multiple of $N_1$ and $N_2$ be $N$.
Let $N = n_1 N_1$, and $N = n_2 N_2$.
We then need to compare $PW_{n_1 P_1}(MARR)$ and $PW_{n_2 P_2}(MARR)$. 
This is the same as to compare

\[ AW_{P_1}(MARR) \text{ and } AW_{P_2}(MARR). \]

**Example:** Consider two projects A and B with MARR=10%

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capital Investment</td>
<td>-$3,500</td>
<td>-$5,000</td>
</tr>
<tr>
<td>Annual Revenue</td>
<td>$1,900</td>
<td>$2,500</td>
</tr>
<tr>
<td>Annual Expenses</td>
<td>-$645</td>
<td>-$1,020</td>
</tr>
<tr>
<td>Useful life</td>
<td>4 years</td>
<td>6 years</td>
</tr>
</tbody>
</table>

The value at the end of useful life is assumed to be zero.
Under the repeatability assumption, we need to compare 3A and 2B. Or, equivalently, their Annual Worth: The AW of A is:

\[-3,500(A/P, 10\%, 4) + 1,900 - 645 = 151\]

and the AW of B is:

\[-5,000(A/P, 10\%, 6) + 2,500 - 1,020 = 332.\]

Therefore, we prefer Project B.

However, not all projects can be repeated. Let us consider a more general setting, when the study period is given. That is, we are given: (1) the MARR; (2) the study period N; (3) the actual project. We want to evaluate whether or not the project is more attractive than the MARR.
**General Approach:**

Let the duration of Project $P$ be $N_p$. Suppose that the study period is $N$.

*Case 1. $N_p < N$.* There are two different possibilities: *(A) $P$ is a service project; (B) $P$ is an investment project.*

In case of *(A)*, one may consider repeating $P$ till $N$ is filled. Then truncate the possible over-time.

In case of *(B)*, one may re-invest the worth of $P$ at the MARR till the end.

*Case 2. $N_p > N$.* Truncate the project at $N$ using an estimated market value of $P$ at $N$. 
Consider the previous example again.

Suppose the study period is changed to 6 year.

Then,

\[
FW(10\%)_A \\
= \left[ -3,500(F/P, 10\%, 4) + (1,900 - 645)(F/A, 10\%, 4) \right](F/P, 10\%, 2) \\
= \$847
\]

and

\[
FW(10\%)_B \\
= -5,000(F/P, 10\%, 6) + (2,500 - 1,020)(F/A, 10\%, 6) \\
= \$2,561
\]

So you still prefer Project B in this case.
**Example:** A company needs to have four additional forklift trucks to support a warehouse, which is anticipated to be shutdown in 8 years. Two mutually exclusive alternatives are identified:

<table>
<thead>
<tr>
<th></th>
<th>Truck 1</th>
<th>Truck 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Capital Investment</strong></td>
<td>-$184K</td>
<td>-$242K</td>
</tr>
<tr>
<td><strong>Annual Expense</strong></td>
<td>-$30K</td>
<td>-$26.7K</td>
</tr>
<tr>
<td><strong>Useful life</strong></td>
<td>5 years</td>
<td>7 years</td>
</tr>
<tr>
<td><strong>Salvage Value</strong></td>
<td>$17K</td>
<td>$21K</td>
</tr>
</tbody>
</table>

The three-year lease cost is $104K per year.

The one-year lease cost is $134K per year.

The MARR of the firm is 15%.

Which type of the truck should the firm buy?
Let us assume that 8 is the study period.

We need to compare the cash flows:

<table>
<thead>
<tr>
<th>Year</th>
<th>Truck 1</th>
<th>Truck 2</th>
<th>Diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year 0</td>
<td>-$184K</td>
<td>-$242K</td>
<td>-$58K</td>
</tr>
<tr>
<td>Year 1</td>
<td>-$30K</td>
<td>-$26.7K</td>
<td>$3.3K</td>
</tr>
<tr>
<td>Year 2</td>
<td>-$30K</td>
<td>-$26.7K</td>
<td>$3.3K</td>
</tr>
<tr>
<td>Year 3</td>
<td>-$30K</td>
<td>-$26.7K</td>
<td>$3.3K</td>
</tr>
<tr>
<td>Year 4</td>
<td>-$30K</td>
<td>-$26.7K</td>
<td>$3.3K</td>
</tr>
<tr>
<td>Year 5</td>
<td>-$13K</td>
<td>-$26.7K</td>
<td>-$13.7K</td>
</tr>
<tr>
<td>Year 6</td>
<td>-$104K</td>
<td>-$26.7K</td>
<td>$77.3K</td>
</tr>
<tr>
<td>Year 7</td>
<td>-$104K</td>
<td>-$5.7K</td>
<td>$98.3K</td>
</tr>
<tr>
<td>Year 8</td>
<td>-$104K</td>
<td>-$134K</td>
<td>-$30K</td>
</tr>
</tbody>
</table>
Since

\[ PW_{Diff}(15\%) = 5,171, \]

so we find Truck 2 more attractive.

One may also apply the ERR method to find

\[ err_{Diff} = 15.97\%, \]

accepting the additional price for Truck 2.
If the study period is shorter than the useful life of an equipment, then it is crucial to estimate the market value of the equipment at that time.

*The Imputed Market Value Technique:*

(implied market value)

Suppose $N < N_p$. Let MARR=$r\%$, the initial capital investment be $C$ and the market value at $N_p$ be $S$. Then,

$$MV_N = \text{‘Capital Recovery’ } \times (P/A, r\%, N_p - N)$$

$$+ S \times (P/F, r\%, N_p - N).$$

In other words, $MV_N$ is the sum of two terms, the first being the PW at year $T$ of the remaining *Capital Recovery value*, and the second is the PW at year $T$ of the *Salvage value*. 

Shuzhong Zhang
**Example:** Suppose that we consider a project with the following data:

- Capital Investment = $47,600
- Useful Life = 9 years
- Salvage Value = $5,000.

Then, the capital recovery amount is

\[ \$47,600(A/P,20\%,9) - \$5,000(A/F,20\%,9) = \$11,569. \]

If we want to consider the end of year 5, then

\[ PW_{CR}(20\%) = \$11,569 \times (P/A,20\%,4) = \$29,949 \]

and

\[ PW_{S}(20\%) = \$5,000(P/F,20\%,4) = \$2,412. \]

Therefore,

\[ MV_5 = \$29,949 + \$2,412 = \$32,361. \]
Exercise: Consider two projects.

<table>
<thead>
<tr>
<th></th>
<th>Project 1</th>
<th>Project 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capital Investment</td>
<td>-$40K</td>
<td>-$60K</td>
</tr>
<tr>
<td>Annual Benefit</td>
<td>$12K</td>
<td>$10K</td>
</tr>
<tr>
<td>Useful life</td>
<td>4 years</td>
<td>8 years</td>
</tr>
<tr>
<td>Salvage Value</td>
<td>$24K</td>
<td>$40K</td>
</tr>
</tbody>
</table>

Suppose that $MARR = 10\%$.

We now wish to compare the projects.

• We assume that the projects can be repeated.

Then, $AW_{P_1}(10\%) = $4,552 and $AW_{P_2}(10\%) = $2,252.

So, Project 1 appears to be more attractive.
• $N = 8$, and suppose that Project 1 cannot be repeated.

In that case we will replace what remains after year 4 by MARR:

$$FW_{P_1}(10\%) = -40K(F/P, 10\%, 8) + [12K(F/A, 10\%, 4) + 24K](F/P, 10\%, 4)$$

$$= 30,933$$

and

$$FW_{P_2}(10\%) = -60K(F/P, 10\%, 8) + 10K(F/A, 10\%, 8) + 40K$$

$$= 25,743.$$ 

Still Project 1 appears to be more attractive.
• Assume that \( N = 4 \).

We need to compute the implied market value of Project 2 at year 4.

According to our formula, this value is

\[
MV_4(P_2) = [60K(A/P, 10\%, 8) - 40K(A/F, 10\%, 8)] \\
\times (P/A, 10\%, 4) + 40K(P/F, 10\%, 4)
\]

\[
= $51,880.
\]

Now we can compute the Annual Worth of Project 2.

\[
AW_{P_2}(10\%, 4 \text{ years})
\]

\[
= -60K(A/P, 10\%, 4) + $10K + $51.88K(A/F, 10\%, 4)
\]

\[
= $2,250.
\]
What happens if the projects are not mutually exclusive?

This leads to the issue of capital budgeting:

\[
\begin{align*}
\text{maximize} & \quad \sum_{i=1}^{n} b_i x_i \\
\text{subject to} & \quad \sum_{i=1}^{n} c_i x_i \leq C \\
& \quad x_i = 0 \text{ or } 1, \ i = 1, \ldots, n.
\end{align*}
\]

*Example:* Budgeting with a total budget of $500,000

Shuzhong Zhang
The optimal solution $x^* = [1, 0, 1, 1, 1, 1, 0]$. 

<table>
<thead>
<tr>
<th>Project</th>
<th>Outlay</th>
<th>PW</th>
<th>BC Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>300</td>
<td>3.00</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>50</td>
<td>2.50</td>
</tr>
<tr>
<td>3</td>
<td>150</td>
<td>350</td>
<td>2.33</td>
</tr>
<tr>
<td>4</td>
<td>50</td>
<td>110</td>
<td>2.20</td>
</tr>
<tr>
<td>5</td>
<td>50</td>
<td>100</td>
<td>2.00</td>
</tr>
<tr>
<td>6</td>
<td>150</td>
<td>250</td>
<td>1.67</td>
</tr>
<tr>
<td>7</td>
<td>150</td>
<td>200</td>
<td>1.33</td>
</tr>
</tbody>
</table>
The budgeting problems may actually involve mutually exclusive projects. Example:

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>Cost</th>
<th>PW</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Road</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Concrete, 2 lanes</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>Concrete, 4 lanes</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>Asphalt, 2 lanes</td>
<td>1.5</td>
<td>3</td>
</tr>
<tr>
<td>Asphalt, 4 lanes</td>
<td>2.2</td>
<td>4.3</td>
</tr>
<tr>
<td><strong>Bridge</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Repair</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>Add lane</td>
<td>1.5</td>
<td>1.5</td>
</tr>
<tr>
<td>New</td>
<td>2.5</td>
<td>2.5</td>
</tr>
<tr>
<td><strong>Traffic</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lights</td>
<td>0.1</td>
<td>0.3</td>
</tr>
<tr>
<td>Lane</td>
<td>0.6</td>
<td>1</td>
</tr>
<tr>
<td>Underpass</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>
This results in another 0-1 integer program:

\[
\begin{align*}
\text{max} & \quad 4x_1 + 5x_2 + 3x_3 + 4.3x_4 + x_5 + 1.5x_6 + 2.5x_7 + 0.3x_8 + x_9 + 2x_{10} \\
\text{s.t.} & \quad 2x_1 + 3x_2 + 1.5x_3 + 2.2x_4 + 0.5x_5 + 1.5x_6 + 2.5x_7 + 0.1x_8 + 0.6x_9 + x_{10} \leq 5 \\
& \quad x_1 + x_2 + x_3 + x_4 \leq 1 \\
& \quad x_5 + x_6 + x_7 \leq 1 \\
& \quad x_8 + x_9 + x_{10} \leq 1 \\
& \quad x_i = 0 \text{ or } 1, \ i = 1, 2, \ldots, 10.
\end{align*}
\]

The Spreadsheet solution says that at the optimality,

\[
x_2^* = 1, \ x_5^* = 1, \ x_{10}^* = 1,
\]

and all other variables are zero.
Break-even Analysis

In actually decision making, it is important to take into account the uncertainty.

One approach is the so-called break-even analysis.

*Example:* Consider the cash flow

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-100M</td>
</tr>
<tr>
<td>1</td>
<td>25.01M</td>
</tr>
<tr>
<td>2</td>
<td>21M</td>
</tr>
<tr>
<td>3</td>
<td>21M</td>
</tr>
<tr>
<td>4</td>
<td>21M</td>
</tr>
<tr>
<td>5</td>
<td>21M</td>
</tr>
</tbody>
</table>

The market value of the project at year 5 is estimated to be 51M.
But this estimation is by no means certain. What do we do?

Suppose that MARR=11%.

Let the market residual value be $S$.

Then,

\[
PW = -100 + \frac{25.01}{1.11} + \frac{21}{1.11^2} \cdot \frac{1 - 1.11^4}{0.11} + \frac{S}{1.11^5}.
\]

From this formula, it is easy to see that

\[
PW \geq 0
\]

if and only if

\[
S \geq 24.4M.
\]

Therefore, any residual value higher than 24.4M will make the project valuable.
Another example of this nature is as follows.

Consider the following project:

| Investment | -$1,000 |
| Annual Return | $400 |
| Duration | ? |
| MARR | 10% |

Let the duration be $N$. Then,

$$PW = -1,000 + 400 \frac{1 - \frac{1}{1.1^N}}{0.1}.$$ 

Hence $PW \geq 0$ if and only if

$$N \geq 3.02 \text{ years}.$$