5. Evaluating a Single Project

If a project is to be undertaken, then it is important to evaluate its worth in monetary terms.

This involves the time dimension of money.

Let us turn everything into the present time.

For simplicity we assume that the MARR is constant. In this respect, the role of the MARR is very much like the interest rate at which the firm should compare all the cash flows against with.
Example: Suppose that your savings account gives you an annual interest rate 6%. You have got $10,000 in that account. A friend of yours needs money urgently. He proposes to borrow your $10,000 now and pay back in the following scheme:

<table>
<thead>
<tr>
<th></th>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$1,000</td>
<td>$5,000</td>
<td>$6,000</td>
</tr>
</tbody>
</table>

Do you want to do it? (assuming there is no default risk, and no moral obligations as well).
The present worth of a project:

\[ PW(i\%) = \sum_{n=0}^{N} F_n (1 + i)^{-n}. \]

**Example:** \( F_0 = -25,000 \), \( F_1 = \cdots = F_4 = 8,000 \), and \( F_5 = 13,000 \). Suppose that the MARR is 20%. Then, the present worth of the project is:

\[
PW(20\%) = 8,000(P/A, 20\%, 5) \\
+ 5,000(P/F, 20\%, 5) - 25,000 \\
= 934.29.
\]
How can we evaluate a project with a sequence of cash flows and a known minimum attractive rate of return (namely MARR)?

Well, this depends on whether the money is well spent. In particular, in the sense of opportunity cost, we need to see whether the project is more attractive than the MARR.

A practical way is to compute the PW of the cash flow under the MARR, and see whether it is positive.
How about discounting everything to the future?

This is the same idea as discounting to the present worth, and it is known as the future worth method.

Formally:

**Present worth method:**

If the present worth of a project under the MARR is positive, then it is acceptable; otherwise not.

**Future worth method:**

If the future worth of a project under the MARR is positive, then it is acceptable; otherwise not.
Example (continue): At present, an equipment costs $25,000 to install. The productivity attributed by the equipment is $8,000 each year for 5 years. In the end of the 5th year it will have a salvage value of $5,000. Suppose that the firm’s MARR is 20% annually. Is this equipment worthy?

Solution:

\[
PW(20\%) = 8,000(P/A, 20\%, 5) + 5,000(P/F, 20\%, 5) - 25,000
\]

\[
= $934.29
\]
Interestingly, the type of the interest rate can be important in this analysis.

Suppose we use the continuous rate for the same problem. Then,

\[
PW(20\%)_c = 8,000(P/A, 20\%, 5)_c + 5,000(P/F, 20\%, 5)_c - 25,000 \\
= 8,000 \times 2.8551 + 5,000 \times 0.3679 - 25,000 \\
= -319.60
\]

It is then *not* economically justified!

The FW method gives:

\[
FW(20\%) = 8,000(F/A, 20\%, 5) + 5,000 - 25,000(F/P, 20\%, 5) \\
= 2,324.90
\]
One may wonder whether or not the PW and FW method will lead to different conclusions.

The answer is: IMPOSSIBLE!

Think about it: Why not?

Another handy method is the *Annual Worth Method*:

If the annual worth of a project under the MARR is positive, then it is acceptable; otherwise not.
All the PW, FW and AW methods will lead to the same conclusion. However, sometimes it can be useful to put things in different perspectives. In particular, the AW method is related to some useful concepts. For instance, the *Capital Recovery Amount* is

\[ CR(i\%) = I(A/P, i\%, N) - S(A/F, i\%, N) \]

where \( I \) is the initial investment, \( S \) is the salvage value at the end. The Annual Worth can be specified as

\[ AW(i\%) = R - E - CR(i\%) \]

where \( R \) is the annual revenue, and \( E \) is the annual expense.
Return from a project can justify its investments if

\[ R \geq E + CR(i\%). \]

The quantity \( E + CR(i\%) \) is defined as the *equivalent uniform annual cost* (EUAC); i.e.

\[ EUAC(i\%) = E + CR(i\%). \]

EUAC is a threshold that a successful project must earn annually *on average*.
Let us examine the previous example again.

The Capital Recovery Amount is

\[
CR(20\%) = 25,000 \times (A/P, 20\%, 5) - 5,000 \times (A/F, 20\%, 5) = 7,687.6
\]

The annual revenue is \( R = 8,000 \). No expense is specified. Hence

\[
EUAC(20\%) = 7,687.6
\]

and

\[
AW(20\%) = 8,000 - 7,687.6 = 312.4
\]
**Example:** A real-estate agency decides to build a 25-unit apartment complex for renting. The costs are as follows:

<table>
<thead>
<tr>
<th>Item</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>Land</td>
<td>$50,000</td>
</tr>
<tr>
<td>Building</td>
<td>$225,000</td>
</tr>
<tr>
<td>Nr of years</td>
<td>20</td>
</tr>
<tr>
<td>Maintenance</td>
<td>$35/p.m.</td>
</tr>
<tr>
<td>Tax</td>
<td>10% of investment</td>
</tr>
</tbody>
</table>

Suppose the occupancy is 90% and the MARR is 12%.

What is the minimum rent to charge?
The costs are:

Initial Investment:

\[ 50,000 + 225,000 = 275,000 \]

Taxes:

\[ 0.1 \times 275,000 = 27,500 \text{ p.a.} \]

Maintenance:

\[ 35 \times 25 \times 12 = 10,500 \]
The annuity of the costs is:

\[ EUAC(20\%) = 275,000(A/P, 12\%, 20) - 50,000(A/F, 12\%, 20) + 27,500 + 10,500 \]

\[ = 74,123 \]

Therefore, the minimum rate should be:

\[ \frac{74,123}{12 \times 25 \times 90\%} = 274.53 \]

per month.
Internal Rate of Return

Let us consider a previous problem again:

<table>
<thead>
<tr>
<th>Year 0</th>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10,000</td>
<td>1,000</td>
<td>5,000</td>
<td>6,000</td>
</tr>
</tbody>
</table>

Suppose now that there is no interest rate (MARR) involved.

You wonder if this cash flow itself is “equivalent to” a kind of interest rate. This can be seen by considering the equation

\[-10K + \frac{1K}{1 + x} + \frac{5K}{(1 + x)^2} + \frac{6K}{(1 + x)^3} = 0\]

Let the solution be \(x = i^*\).

This is a kind of intrinsic, internally generated rate of return.

In our case, \(i^* = 7.89\%\).

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Definition of *Internal Rate of Return*:

Let $R_n$ be the revenue and $E_n$ be the expense on year $n$.

The $irr$ is defined to be the rate such that

$$PW(IRR) = \sum_{n=0}^{N} R_n(P/F, irr\%, n) - \sum_{n=0}^{N} E_n(P/F, irr\%, n) = 0.$$ 

Note: There might be more than one IRR!
Example: Consider the equipment problem once more. The $irr$ should satisfy

$$8,000(P/A, irr\%, 5) + 5,000(P/F, irr\%, 5) - 25,000 = 0$$

Let $irr$ be $x$. The equation is actually

$$8,000\frac{(1 + x)^5 - 1}{x(1 + x)^5} + 5,000\frac{1}{(1 + x)^5} = 25,000$$

A trial and error method gives the solution:

$$irr \approx 21.577\%$$

So if the MARR is 20%, then installing the equipment is economically justified.
A bond’s yield is defined by its IRR of the promised cash flow.

Remember we have the specifications of a bond:

- Face value $Z$
- Disposal price $C$ (most of the time, $C = Z$)
- Coupon rate $r$
- Yield $y$

Then, the yield $y$ satisfies the equation

$$B - C(P/F, i\%, N) - rZ(P/A, i\%, N) = 0$$

where $B$ is the current price of the bond.

It is particularly simple to calculate the yield of a zero-coupon bond

$$y = \left(\frac{C}{B}\right)^{\frac{1}{N}} - 1.$$
Example: “A 5.6% loan plan”.

I was offered a loan with 5.6% annual rate for three years by a bank when I wished to purchase a car in Hong Kong (August 1999). The price for the car was HK$115,527, and I was told that the monthly payment is HK$3,749.

Questions:

1) How is this rate calculated?

2) What is the real monthly rate?
1) \[ 115,527 \times (0.056/12 + 1/36) \approx 3,749 \]

2) The equation to be considered is:\[ 3,749(P/A, irr\%, 36) = 115,527 \]
Let $i_{\text{rr}}$ be $x$. This result in an equation

$$
\frac{(1 + x)^{36} - 1}{x(1 + x)^{36}} = \frac{115,527}{3,749}
$$

The root is $x^* = 0.0087$. So the effective annual rate is:

$$
i_{\text{rr}} = (1 + 0.0087)^{12} - 1 = 0.1095!
$$
Example: A small company needs to borrow $160,000. The local banker makes the following statement: “We can loan you $160,000 at a very favorable rate of 12% per year for 5 years. However, to ensure this loan you must agree to establish a checking account in which the minimum average balance is $32,000. In addition, your interest payments are due at the end of each year and the principal will be repaid in a lump-sum amount at the end of year five”. What is the effective annual interest rate being charged?
The cash flow is:

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year 0</td>
<td>$160K-$32K</td>
</tr>
<tr>
<td>Year 1 to 4</td>
<td>$12% \times $160K = $19.2K</td>
</tr>
<tr>
<td>Year 5</td>
<td>$19.2K+$128K</td>
</tr>
</tbody>
</table>

The IRR of the above cash flow turns out to be 15\%.

Moral: Be careful with the words on the contract!
The IRR method has at least two drawbacks.

1) *There can be multiple solutions.*

Consider an investment with initial capital 1M, 5M return in one year and 5M payment in the end of second year. By the IRR method one needs to solve the following equation:

\[-1 + 5 \times (1 + irr)^{-1} - 5 \times (1 + irr)^{-2} = 0\]

This leads to two solutions:

\[irr \approx 0.382 \text{ or } irr \approx 2.618\]

The correct answer needs to be further decided.

The multiple solutions can only occur when the signs of the cash flow have changed more than once.
Let us see why the IRR is uniquely defined for a pure investment project (or a pure debt cash flow).

The cash flow of an investment project:

$$-I_0, R_1, \ldots, R_N$$

where $I_0 > 0$ and $R_n > 0$, $n = 1, \ldots, N$.

The IRR is the root of the following equation

$$-I_0 + \sum_{n=1}^{N} \frac{R_n}{(1 + x)^n} = 0.$$

Now the function

$$-I_0 + \sum_{n=1}^{N} \frac{R_n}{(1 + x)^n}$$

is decreasing in $x$. So it can have at most one root. (Why?)
2) *It might be impossible to realize the IRR!*

A crucial assumption underlying the IRR method is that the revenues can be reinvested at the IRR. But this may not be possible.

One way to circumvent the difficulty is to distinguish between the revenues and the expenses.

This approach is called the *External Rate of Return* method.
**External Rate of Return:**

Let $r$ be some external investment return rate available to the firm. Let $R_n$ be the revenue and $E_n$ be the expense in year $n$. Then, the *external rate of return*, $err$, is defined to be the root of the following equation:

$$
\sum_{n=0}^{N} R_n(F/P, r\%, N - n) = \sum_{n=0}^{N} E_n(P/F, r\%, n)(F/P, err\%, N)
$$

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This leads to:

\[(1 + err)^N = \frac{\sum_{n=0}^{N} R_n(F/P, r\%, N - n)}{\sum_{n=0}^{N} E_n(P/F, r\%, n)}\]

Therefore, the ERR method always leads to a unique rate with \(1 + err \geq 0\).

\textit{Example:} If we examine the equipment investment problem in this view (using \(r = MARR\)), then

\[(1 + err)^5 = \frac{64,532.8}{25,000} = 2.5813\]

Therefore \(err = 20.88\%\).
The ERR can be understood by the following input/output diagram:
You need
\[
\sum_{n=0}^{N} \frac{E_n}{(1 + r)^n}
\]
in Year 0 as input, in order to cover all the negative cash flows.
At the end of Year \( N \), the face value of all the accumulated positive cash flow (output) equals
\[
\sum_{n=0}^{N} R_n (1 + r)^{N-n}
\]
Hence the rate of return in this input/output perspective is
\[
\left( \frac{\sum_{n=0}^{N} R_n (1 + r)^{N-n}}{\sum_{n=0}^{N} \frac{E_n}{(1+r)^n}} \right)^{1/N}
\]
Look back at the loan problem once more.

The input is $100,000 right now (Year 0).

The output at Year 3 is:

\[ 10K \times (1 + 6\%)^2 + 50K \times (1 + 6\%) + 60K \]
\[ \approx 124.24K \]

This amounts to

\[ \text{err} = \left( \frac{10(1.06)^2 + 50(1.06) + 60}{100} \right)^{1/3} - 1 \]
\[ = 7.5\%. \]
How can we use the IRR and ERR to evaluate a project?

The IRR Method

If $irr > MARR$ then the project is acceptable, o.w. not.

In case one gets more than one $irr$, or, there is indeed an alternative investment possibility available, then we can use:

The ERR Method

If $err > MARR$ then the project is acceptable, o.w. not.

If $r = MARR$, which happens in most cases, then the ERR and the IRR methods lead to the same conclusion.
The Payback Period Method.

In financial decision making, it is often desirable to have a high degree of liquidity. One measurement is the payback period method.

Assume that the initial investment is \( I \) and is made at the beginning only.

**Simple Payback:** The minimal \( \theta \) such that

\[
\sum_{n=1}^{\theta} (R_n - E_n) - I \geq 0
\]

For the equipment problem, the simple payback period is 4 years.

**Discounted Payback:** The minimal \( \theta \) such that

\[
\sum_{n=1}^{\theta} (R_n - E_n)(P/F, i\%, n) - I \geq 0
\]

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