Chapter 5. Applied Interest Rate Analysis

Capital Budgeting

Budgeting with a total budget of $500K, with the following data:

Project	Outlay	NPV	BC Ratio
1	100	300	3.00
2	20	50	2.50
3	150	350	2.33
4	50	110	2.20
5	50	100	2.00
6	150	250	1.67
7	150	200	1.33

The problem can be modeled with 0-1 integer programming as follows

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{n} b_i x_i \\ \text{subject to} & \sum_{i=1}^{n} c_i x_i \le C \\ & x_i = 0 \text{ or } 1,\ i = 1, ..., n. \end{array}$$

Solving 0-1 integer programming problems can be done using Excel.

For our particular example, the optimal solution is found to be $x^* = [1, 0, 1, 1, 1, 1, 0]$.

Alternatives	Cost	NPV
Road		
Concrete, 2 lanes	2	4
Concrete, 4 lanes	3	5
Asphalt, 2 lanes	1.5	3
Asphalt, 4 lanes	2.2	4.3
Bridge		
Repair	0.5	1
Add lane	1.5	1.5
New	2.5	2.5
Traffic		
Lights	0.1	0.3
Lane	0.6	1
Underpass	1	2

This results in another 0-1 integer program:

$$\begin{aligned}
\max \quad & 4x_1 + 5x_2 + 3x_3 + 4.3x_4 + x_5 + 1.5x_6 + 2.5x_7 + 0.3x_8 + x_9 + 2x_{10} \\
\text{s.t.} \quad & 2x_1 + 3x_2 + 1.5x_3 + 2.2x_4 + 0.5x_5 \\
& \quad +1.5x_6 + 2.5x_7 + 0.1x_8 + 0.6x_9 + x_{10} \le 5 \\
& x_1 + x_2 + x_3 + x_4 \le 1 \\
& x_5 + x_6 + x_7 \le 1 \\
& x_8 + x_9 + x_{10} \le 1 \\
& x_i = 0 \text{ or } 1,\ i = 1, 2, ..., 10.
\end{aligned}$$

The Spreadsheet solution says that at the optimality,

$$x_2^* = 1,\ x_5^* = 1,\ x_{10}^* = 1,$$

and all other variables are zero.

Optimal Portfolios

The Cash Matching Problem.

Suppose that one wishes to use bonds to match the required cash. There are 10 bonds available for this purpose, each with different maturity dates. The specifications are as follows.

1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
10	7	8	6	7	5	10	8	7	100
10	7	8	6	7	5	10	8	107	
10	7	8	6	7	5	110	108		
10	7	8	6	7	105				
10	7	8	106	107					
110	107	108							
109	94.8	99.5	93.1	97.2	92.9	110	104	102	95.2

The cash requirements on each year are

Year	1	2	3	4	5	6
Cash	100	200	800	100	800	1,200

The problem can be formulated as a linear program:

$$\begin{array}{ll} \text{minimize} & \sum_{j=1}^{n} p_j x_j \\ \text{subject to} & \sum_{j=1}^{n} c_{ij} x_j \geq r_i,\ i = 1, ..., m \\ & x_j \geq 0,\ j = 1, ..., n. \end{array}$$

Using Excel or Matlab we can solve the above problem and get a solution as:

$$x_2^* = 11.2,\ x_4^* = 6.81,\ x_8^* = 6.3,\ x_9^* = 0.28$$

and all other values are zero.

The actual cash fulfillment is

Year	1	2	3	4	5	6
Need	100	200	800	100	800	1,200
Actual	171.7	200	800	119.3	800	1,200

Dynamic Cash Flow Procedure

Any actions will inevitably lead to consequences and reactions. So is true for cash flow management.

To understand what can happen in the context of cash flow management, it is quite natural to introduce graphic representations.

Since cash is one-dimensional, and the time is also one-dimensional, one can simply draw a *tree*-like graph: the x-axis represents the time; the y-axis represents the cash; each node represents a decision point.

Typically, it is common to use the binomial tree representations; the binomial lattice representations; the trinomial trees or lattices.

Dynamic Management

Dynamic programming is a method to help making decisions at different time steps, by analyzing the problem from the end to the beginning.

The key idea is to recognize that a part of an optimal solution for a problem is itself also an optimal solution for this part of the problem.

We shall only illustrate this by considering a problem using a binomial tree or binomial lattice to represent decisions and the consequences over time.

A node (k, i) is used to represent node i at stage k. Departing from i there are two follow-up nodes, call them $C(i)$.

Suppose that the optimal value at node (k, i) is $V_{k,i}$. The cash flow (benefit) to move from (k, i) to $(k+1, j)$ where $j \in C(i)$ is c_{ki}^{j}.

Then we have a recursive formula:

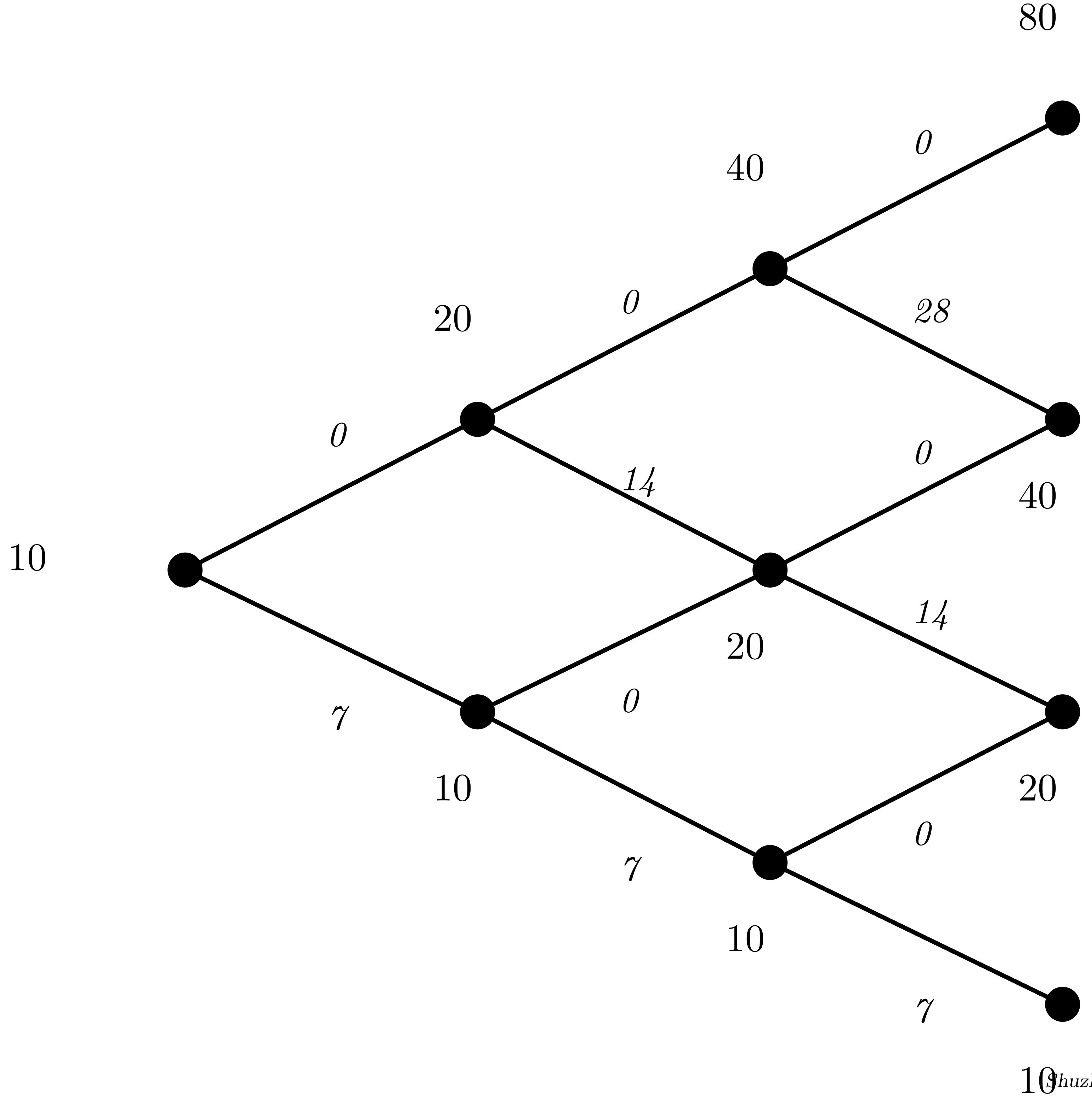
$$V_{k,i} = \max_{j \in C(i)} \{c_{ki}^{j} + d_k V_{k+1,j}\}.$$

Example 5.4. Resource and profit

Suppose that you own a certain resource which can be regenerated. Each period, you may decide to use the resource or let it be regenerated. If you let the resource be at rest, then its amount will double in the next period. Otherwise, you may decide to profit from the resource. If you do so, then you take away 70% of all the available resource, and the resource will be regenerated in the next season to the current level (before the usage).

Suppose that initially there are 10 units of resource available. Making use of each unit of the resource will bring you a profit of $1K. The discount rate is 80% for each period.

The question is: What is the optimal management policy?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The optimal policy can be evaluated by the dynamic programming principle.

Construct a graph for this problem. First, we assume $t = 3$ is the end of the horizon. Therefore,

$$V_{3,1} = 0,\ V_{3,2} = 0,\ V_{3,3} = 0,\ V_{3,4} = 0.$$

Now,

$$\begin{aligned} V_{2,1} &= \max\{0 + 0.8 \times V_{3,1}, 28 + 0.8 \times V_{3,2}\} \\ &= 28 \\ V_{2,2} &= \max\{0 + 0.8 \times V_{3,2}, 14 + 0.8 \times V_{3,3}\} \\ &= 14 \\ V_{2,3} &= \max\{0 + 0.8 \times V_{3,3}, 7 + 0.8 \times V_{3,4}\} \\ &= 7. \end{aligned}$$

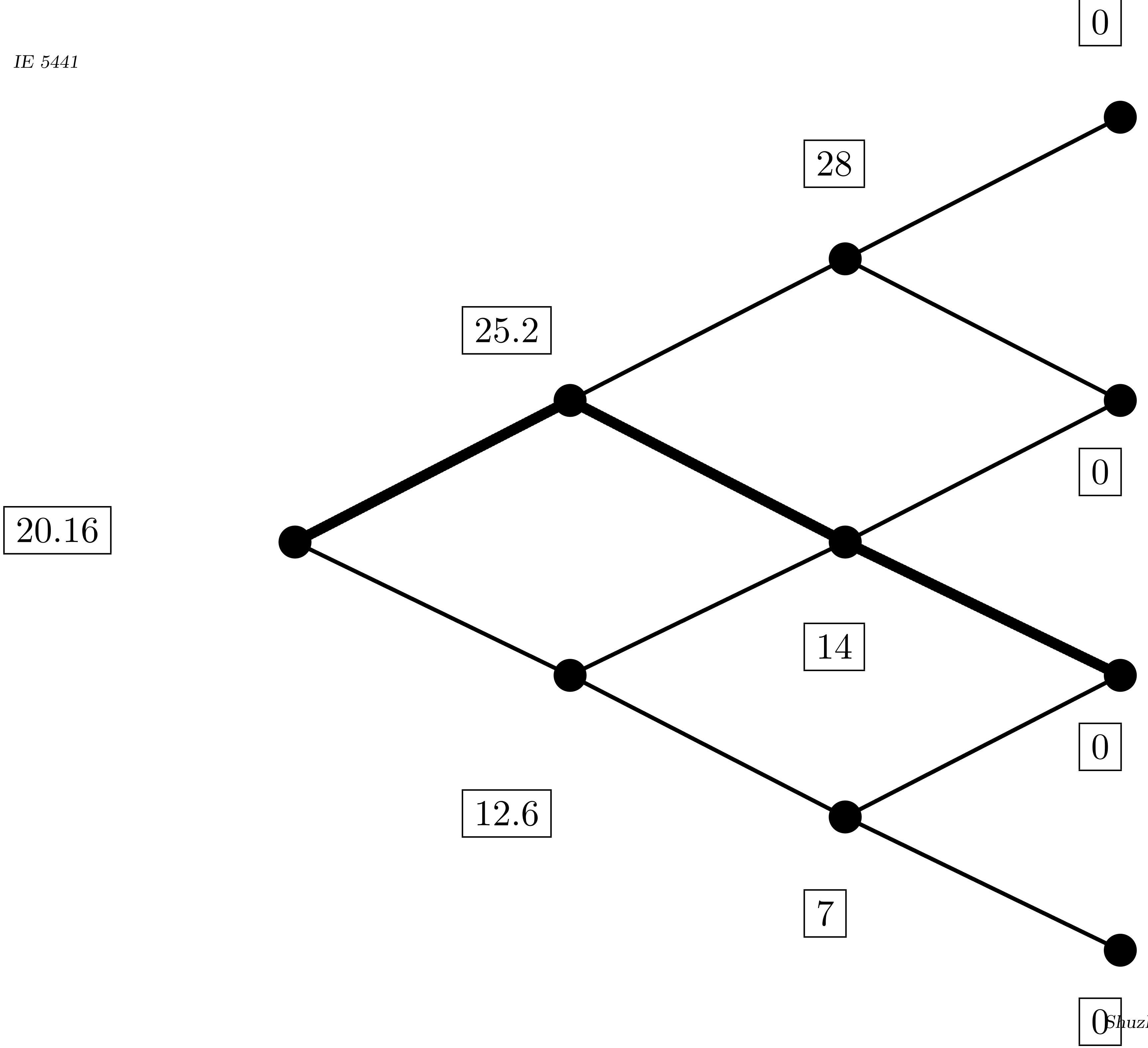
Take one step back, we have

$$\begin{aligned} V_{1,1} &= \max\{0 + 0.8 \times V_{2,1}, 14 + 0.8 \times V_{2,2}\} \\ &= 25.2 \\ V_{1,2} &= \max\{0 + 0.8 \times V_{2,2}, 7 + 0.8 \times V_{2,3}\} \\ &= 12.6. \end{aligned}$$

Finally,

$$\begin{aligned} V_{0,1} &= \max\{0 + 0.8 \times V_{1,1}, 7 + 0.8 \times V_{1,2}\} \\ &= 20.16. \end{aligned}$$

The solution can be presented by a graph.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25.2
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20.16
14
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12.6
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The idea of *dynamic programming* can be successfully applied to the case where the decisions may be infinite at each point.

Example 5.5. (Complexico mine) The Complexico mine is for lease. It is increasingly difficult now to extract gold from the mine. Let x be the amount of gold at the beginning of a year, the cost to extract z $(0 \leq z \leq x)$ out of the mine is $\$500z^2/x$. Suppose that the current amount of gold remaining is $x_0 = 50,000$ ounces, and the price is \$400 per ounce. We are considering the purchase of a 10-year lease. The interest is 10%. How much is this lease cost?

Let us consider the situation in Year 9. Suppose we have x_9 left in the mine then. The optimal decision will be

$$V_9(x_9) = \max_{0 \le z_9 \le x_9} \left(g z_9 - 500 z_9^2 / x_9 \right) = \frac{g^2}{2000} x_9 =: K_9 x_9$$

where g is the price of gold.

The problem in Year 8 is:

$$\begin{aligned} V_8(x_8) &= \max_{0 \le z_8 \le x_8} \left(g z_8 - 500 z_8^2 / x_8 + d K_9 (x_8 - z_8) \right) \\ &= \left[\frac{(g - d K_9)^2}{2000} + d K_9 \right] x_8 =: K_8 x_8, \end{aligned}$$

where $0 < d < 1$ is the discounting factor. In our case we set $d = 1/1.1$.

Recursively we get

$$K_j = \frac{(g - dK_{j+1})^2}{2000} + dK_{j+1},\ j = 8, 7, 6, 5, 4, 3, 2, 1, 0.$$

In our particular setting, $g = 400$ and $d = 1/1.1$. Therefore, we can compute the K_j values (in reverse order):

$K_j = 80.00,\ 126.3,\ 155.5,\ 174.8,\ 188.0,\ 197.1,\ 203.6,\ 208.2,\ 211.5,\ 213.8$

This shows that $V_0(x_0) = 213.8 \times x_0$. The answer to our question is therefore

$$V_0(50,000) = 213.8 \times 50,000 = \$10,690,000.$$

The Harmony Theorem

Suppose that we have a set of investment alternatives. The cash reward is represented by the y-axis, and the cost is represented by the x-axis. We then see a set of points in the first orthant $P_1(x_1, y_1), P_2(x_2, y_2), \cdots, P_n(x_n, y_n)$. The interest rate available to us is r.

According to the NPV principle, we should compare $PV_i = -x_i + y_i/(1 + r)$, $i = 1, 2, ..., n$, and pick the one with the biggest PV.

According to the maximum rate of return principle, it appears that the highest y_i/x_i should be most attractive. *A conflict of thoughts here!*

This can be resolved by imagining that you are taking over the ownership of the projects.

For the current owner of P_i, its value is really $PV_i = -x_i + y_i/(1+r)$. Let us say that PV_1 is maximum.

Now, if you are to take P_1 over, then the current owner will charge you PV_1. Therefore, your *eventual* rate of return for project j becomes:

$$\frac{y_j}{x_j + PV_1} \le \frac{y_1}{x_1 + PV_1} = 1 + r, \text{ for } j \neq 1.$$

In this perspective, the optimal choose is still P_1, and so the conflict is resolved. This is known as the harmony theorem.

Valuation of a Firm

In financial economics, one of the most important problems is to evaluate a firm. At a philosophical level, this is mission impossible. However, some rough estimation can still be done.

Dividend Discount Models

Suppose that D_i is the future dividend payments. Then, one may argue that the value of the firm is

$$V_0 = \sum_{n=1}^{\infty} \frac{D_n}{(1+r)^n}.$$

Let us further assume that $D_{n+1} = D_n(1+g)$. Then

$$V_0 = \frac{(1+g)D_0}{r-g},$$

where D_0 is the current dividend payment.

Example 5.6. The XX Corporation has just paid a dividend of \$1.37M. The company is expected to grow at 10% for the foreseeable future, and hence most analysts project a similar growth in dividends. The discount rate used for the company is 15%. What is the value of the company?

Well, according to our analysis before, it should be

$$V_0 = \frac{1.3M \times 1.10}{0.15 - 0.10} = \$30,140,000.$$

Free Cash Flow

In general, dividend payments do not entirely represent the financial status of the firm. One may wish to take the earnings of the firm into account.

Suppose that Y_n be the gross earnings of the firm in Year n; a portion u is revested to sustain the earning in the future; the growth rate of earning for next year is $g(u)$; the rate of depreciation is α; the capital of the firm is C_n. We have

$$Y_{n+1} = [1 + g(u)]Y_n \text{ and } C_{n+1} = (1 - \alpha)C_n + uY_n.$$

Therefore,

$$C_n = (1 - \alpha)^n C_0 + uY_0 \left\{ \frac{-(1 - \alpha)^n + [1 + g(u)]^n}{g(u) + \alpha} \right\} \approx \frac{uY_0[1 + g(u)]^n}{g(u) + \alpha}.$$

Income Statement

Before tax cash flow	Y_n
Depreciation	αC_n
Taxable income	$Y_n - \alpha C_n$
Income tax (34%)	$0.34(Y_n - \alpha C_n)$
After tax income	$0.66(Y_n - \alpha C_n)$
After tax cash flow	$0.66(Y_n - \alpha C_n) + \alpha C_n$
Investment	uY_n
Free cash flow	$0.66(Y_n - \alpha C_n) + \alpha C_n - uY_n$

Therefore,

$$FCF = \left[0.66 + 0.34\frac{\alpha u}{g(u)+\alpha} - u\right][1+g(u)]^n Y_0.$$

This gives us

$$PV = \left[0.66 + 0.34\frac{\alpha u}{g(u)+\alpha} - u\right]\frac{1+r}{r-g(u)} Y_0.$$

In fact, this can even give us a clue how to optimize the investment rate.

Example 5.8. The XX Corporate has current earning Y_0=\$10 Million, and C_0=\$19.8 Million. The interest rate is 15%. The depreciation rate is $\alpha = 0.1$. The relationship between investment and the growth is $g(u) = 0.12 \left[1 - e^{5(\alpha - u)}\right]$.

Maximizing PV we get $u = 37.7\%$, and $g(u) = 9\%$, with PV=\$67 Million.