Chapter 4. The Term Structure of Interest Rates
The yield curve and term structure

Spot rates:

- Yearly: \((1 + s_t)^t\);
- \(m\) periods per year: \((1 + s_t/m)^{mt}\);
- Continuous: \(e^{stt}\).

Discount Factors:

- Yearly:

\[
d_k = \frac{1}{(1 + s_k)^k};
\]

- \(m\) periods per year:

\[
d_k = \frac{1}{(1 + s_k/m)^{mk}};
\]

- Continuous: \(d_t = e^{-stt}\).
Determining the spot rate

Bond A is a 10-year bond with a 10% coupon. Its price is $P_A = 98.72$. Bond B is a 10-year bond with 8% coupon. Its price is $P_B = 85.89$. Both bonds have face value $100$.

Consider a portfolio with $-0.8$ unit of Bond A and $1$ unit of Bond B. This portfolio will have a face value of $20$, and price $P = P_B - 0.8 \cdot P_A = 6.914$. This portfolio becomes a zero-coupon bond with the 10-year spot rate $s_{10} = (20/P)^{1/10} - 1 = 11.2\%$.
Forward Rates

Let $f$ be the interest rate to borrow money from Year 1 to Year 2 (agreed today).

By the principle of no-arbitrage, we have

$$f = \frac{(1 + s_2)^2}{1 + s_1} - 1.$$

In general, for any $i < j$:

$$(1 + s_j)^j = (1 + s_i)^i (1 + f_{i,j})^{j-i}.$$

Hence,

$$f_{i,j} = \left[ \frac{(1 + s_j)^j}{(1 + s_i)^i} \right]^{1/(j-i)} - 1.$$
In the case of $m$-periods per year:

$$(1 + s_j/m)^j = (1 + s_i/m)^i(1 + f_{i,j}/m)^{j-i},$$

and

$$f_{i,j} = m \left[ \frac{(1 + s_j/m)^j}{(1 + s_i/m)^i} \right]^{1/(j-i)} - m.$$

In the case of continuous compounding:

$$e^{s_{t_2}t_2} = e^{s_{t_1}t_1} e^{f_{t_1,t_2}(t_2-t_1)},$$

and so

$$f_{t_1,t_2} = \frac{s_{t_2}t_2 - s_{t_1}t_1}{t_2 - t_1}.$$
Term structure explanations

*Expectations Theory*

The so-called expectation hypothesis assumes that the market as a whole has an implied expectation regarding the trend of the spot rate in the future (the forward rate). The theory therefore hypothesizes that the market expects the spot rate in the future will typically rise, in the face of inflation.

The downside of the theory is that, as a conclusion of the theory, it seems to be that the market always expects the rates to rise, but this is certainly not always the case!
Liquidity Preference

Another possible explanation for the existence of a non-flat term structure of the interest rates is that the short-term rates are preferred because they appear to be more liquid. This kind of preferences creates a non-flat term structure.

Another factor that is often taken in this explanation is that the longer-term bonds being more sensitive to the change of interest rates are compensated by the market with more attractive prices.
Market Segmentation

This theory attempts to explain the existence of different interest rates by arguing that the needs for different terms are completely independent. Therefore they fluctuate more or less independently as well.

This explanation does not explain though why there is typically an upward trend regarding the term structure.

One may believe that there are some points in each of these explanations, and it may not be one and only one truth behind the myth of interest rate term structures.
Expectation Dynamics

Spot rate forecast

The spot rates are in principle all implied by the following expression:

\[ f_{1,j} = \left[ \frac{(1 + s_j)^j}{1 + s_1} \right]^{1/(j-1)} - 1, \quad j = 2, 3, ..., n \]

These rates become the spot rates at time \( t = 1 \). Letting \( s'_j := f_{1,j} \), we can repeat the above computation to get \( f_{2,j} \) for \( j = 3, 4, ..., n \), and so on and so forth.

This is termed expectations dynamics.
In general, we have $f_{0,j} = s_j =: s_j^0$, $j = 1, 2, ..., n$. Then,

$$s_{j-1}^1 := f_{1,j} = \left[ \frac{(1 + s_j^0)^j}{1 + s_1^0} \right]^{1/(j-1)} - 1, \; j = 2, 3, ..., n.$$ 

Recursively, $s_k^i = f_{i,i+k}$, and

$$(1 + s_{k+1}^{i-1})^{k+1} = (1 + s_1^{i-1})(1 + f_{i,i+k})^k.$$ 

Therefore,

$$s_k^i = \left( \frac{(1 + s_{k+1}^{i-1})^{k+1}}{1 + s_1^{i-1}} \right)^{1/k} - 1.$$
Discount Factors

The discount factor between periods $i$ and $j$ ($j > i$) is:

$$d_{i,j} = \left[ \frac{1}{1 + f_{i,j}} \right]^{j-i}.$$

Under the no-arbitrage assumption, we have

$$d_{i,k} = d_{i,j} d_{j,k} \text{ for any } i < j < k.$$
**Short Rates**

The short rates are the one-period forward rates in the future: $r_k = f_{k,k+1}$. The short rates can be computed recursively by:

$$(1 + s_k)^k = (1 + r_0)(1 + r_1) \cdots (1 + r_{k-1}).$$

Moreover,

$$(1 + f_{i,j})^{j-i} = (1 + r_i)(1 + r_{i+1}) \cdots (1 + r_{j-1}).$$

**Invariance Theorem:** Suppose the interest rates evolve precisely according to the expectation dynamics. Then $1$ invested in this scheme will grow to $(1 + s_n)^n$ regardless how the investment is done.  

*Shuzhong Zhang*
Running Present Value

Under the framework of the expectation dynamics, the present value of a cash flow can be computed in a recursive manner.

Consider a cash flow \((x_0, x_1, \cdots, x_n)\). Its present value is

\[
PV(0) = x_0 + d_{0,1}x_1 + \cdots + d_{0,n}x_n.
\]

Denote

\[
PV(k) = x_k + d_{k,k+1}x_{k+1} + \cdots + d_{k,n}x_n.
\]

We have

\[
PV(k) = x_k + d_{k,k+1}PV(k + 1).
\]

(Question: Any intuitive way to remember this rule?)
Example 4.6. Suppose that the spot rate curve is flat with $s_k = r$ for all $k = 1, 2, ..., n$. Suppose that the cash flow is $(x_1, ..., x_n)$. Then, $PV(n) = x_n$, and

$$PV(k) = x_k + \frac{1}{1+r} PV(k + 1).$$

Example 4.7:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<tbody>
<tr>
<td>CF</td>
<td>20</td>
<td>25</td>
<td>30</td>
<td>35</td>
<td>40</td>
<td>30</td>
<td>20</td>
<td>10</td>
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<td>Discount</td>
<td>.943</td>
<td>.935</td>
<td>.93</td>
<td>.926</td>
<td>.923</td>
<td>.921</td>
<td>.917</td>
<td>1</td>
</tr>
<tr>
<td>PV</td>
<td>168.95</td>
<td>157.96</td>
<td>142.2</td>
<td>120.64</td>
<td>92.49</td>
<td>56.87</td>
<td>29.17</td>
<td>10</td>
</tr>
</tbody>
</table>

Shuzhong Zhang
**Floating Rate Bond**

A floating rate bond has a fixed face value, but the coupon payments are tied to the current (short) rates of interest. After each coupon payment, the time point is called a reset point, and the coupon rate for the next period is set equal to the spot rate.

*Theorem of floating rate value:* The present value of a floating rate bond is equal to its face value at any reset point (after the coupon payment).

At the time of maturity, the payment is: $C + Cr_n$. Viewed in period $n - 1$, this has a present value (discounted by $r_n$) equal to $C$.

In general, after the coupon payment at $k$, we have

$$PV(k) = d_{k,k+1}(C \cdot r_k + PV(k + 1)).$$

Now, $r_k = f_{k,k+1}$, and $d_{k,k+1} = \frac{1}{1+f_{k,k+1}}$. If $PV(k + 1) = C$, then $PV(k) = C$, which is held throughout the process.
Duration

The notion of duration we discussed earlier did not involve a term structure. Now we see how to incorporate the term structure in duration, and how to use the concept to form an immunized bond portfolio.

For a cash flow \((x_{t_0}, x_{t_1}, \cdots, x_{t_n})\), and the spot rates \((s_{t_0}, s_{t_1}, \cdots, s_{t_n})\), its present value is

\[
P V = \sum_{i=0}^{n} x_{t_i} e^{-s_{it_i}}.
\]

Its Fisher-Weil duration is defined as

\[
D_{FW} = \frac{1}{PV} \sum_{i=0}^{n} t_i x_{t_i} e^{-s_{it_i}}.
\]
We wish to find a plan to manage the risk of a *spot rate curve shifting*; that is, the entire curve of the interest rates has a uniform change of $\lambda$.

Introduce

$$PV(\lambda) = \sum_{t=0}^{n} x_{t_i} e^{-(s_i+\lambda)t_i}.$$ 

We have

$$D_{FW} = -\frac{PV'(0)}{PV(0)}.$$
In the case of discrete-time compounding, we have

\[ P(\lambda) = \sum_{k=0}^{n} \frac{x_k}{(1 + s_k/m)^k} \cdot \]

Hence

\[ P'(0) = -\sum_{k=1}^{n} \frac{kx_k/m}{(1 + s_k/m)^{k+1}}. \]

The so-called \textit{quasi-modified duration} is

\[
D_Q = -\frac{P'(0)}{P(0)} = \frac{\sum_{k=1}^{n} (kx_k/m)/(1 + s_k/m)^{k+1}}{\sum_{k=0}^{n} x_k/(1 + s_k/m)^k} \cdot \\
= \frac{1}{PV} \sum_{k=1}^{n} \left( \frac{k}{m} x_k \right) \left( 1 + \frac{s_k}{m} \right)^{-1}(k+1).
\]
**Immunization**

Suppose we have $1$ Million obligation to be paid in 5 years. We consider to buy two bonds to prepare for this obligation. Bond 1 is a 12-year 6% bond with price 65.95, and Bond 2 is 5-year 10% bond with price 101.66.

The PV for Bond 1 is 65.95, and for Bond 2 is 101.66. The modified duration for Bond 1 is 7.07, and is 3.80 for Bond 2. The PV for the obligation is $627,903.01$, and the corresponding quasi-modified duration is 4.56.

Solving

\[
PV = P_1 x_1 + P_2 x_2
\]

\[
PV \times D = P_1 D_1 x_1 + P_2 D_2 x_2.
\]

This leads to \(x_1 = 2,208.17\) and \(x_2 = 4,744.03\).
Numerical simulation:

<table>
<thead>
<tr>
<th>Interest rate change</th>
<th>0</th>
<th>1%</th>
<th>-1%</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Bond 1</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Shares</td>
<td>2,208.00</td>
<td>2,208.00</td>
<td>2,208.00</td>
</tr>
<tr>
<td>Price</td>
<td>65.94</td>
<td>51.00</td>
<td>70.84</td>
</tr>
<tr>
<td>Value</td>
<td>145,602.14</td>
<td>135,805.94</td>
<td>156,420.00</td>
</tr>
<tr>
<td><strong>Bond 2</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Shares</td>
<td>4,744.00</td>
<td>4,744.00</td>
<td>4,744.00</td>
</tr>
<tr>
<td>Price</td>
<td>101.65</td>
<td>97.89</td>
<td>105.62</td>
</tr>
<tr>
<td>Value</td>
<td>482,248.51</td>
<td>464,392.47</td>
<td>501,042.18</td>
</tr>
<tr>
<td><strong>Obligation value</strong></td>
<td>627,903.01</td>
<td>600,063.63</td>
<td>657,306.77</td>
</tr>
<tr>
<td><strong>Difference:</strong></td>
<td>-$52.37</td>
<td>$134.78</td>
<td>$155.40</td>
</tr>
</tbody>
</table>