Chapter 2. Basic Theory of Interest
Interest rates:

- Simple interest;
- Compound interest;
- Nominal interest rate;
- Effective interest rate;
- Continuous time compounding.
Money: Note of value promised to be delivered at different points in time and quantity.

Cash Flow: A prescribed sequence of money/cash exchanges in time.

Financial Instruments: Bills, notes, bonds, futures, contracts, ...

Financial Securities: Well developed financial instruments for trading.

Fixed Income Securities: Securities that promise a fixed income to the holder over a span of time.

Financial Market: Market place where the financial securities are traded.
• Ideal Bank.
• Present value of a cash flow.
• How to evaluate a cash flow with a given interest rate?
• Equivalence between different cash flows.
Comparing two cash flows: *the method of comparing Net Present Value*.

Recall the example we considered before:

Suppose that your savings account gives you an annual interest rate 6%. You have got $100,000 in that account. A friend of yours needs some money urgently. He proposes to borrow your $100,000 now and pay back in the following scheme:

<table>
<thead>
<tr>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10,000</td>
<td>$50,000</td>
<td>$60,000</td>
</tr>
</tbody>
</table>
This can be generalized to the evaluation of a cash flow:

<table>
<thead>
<tr>
<th>Year 0</th>
<th>Year 1</th>
<th>⋮</th>
<th>Year n</th>
<th>⋮</th>
<th>Year N</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-P$</td>
<td>$C_1$</td>
<td>⋮</td>
<td>$C_n$</td>
<td>⋮</td>
<td>$C_N$</td>
</tr>
</tbody>
</table>

Given the interest rate $i$, this boils down to computing

$$NPV := -P + \sum_{n=1}^{N} \frac{C_n}{(1 + i)^n}.$$  

If it is positive then the project is preferred over $i$; o.w., it is not.
Interest rate formulas for

\((P/F, i\%, N), (A/P, i\%, N), (G/P, i\%, N), \ldots\)

Formula \((F/P, i\%, N)\) (Given \(P\) find \(F\)).

The interest for period \(n\) will be \(i(1 + i)^{n-1}, i = 1, \ldots, N\). So, at the end of year \(N\), the total amount of interests will be

\[
1 \times i \times \sum_{n=1}^{N} (1 + i)^{n-1} = (1 + i)^N - 1.
\]

Now, the principal amount, $1, has to be paid as well. So, the total equivalent amount is

\((1 + i)^N;\)

or,

\((F/P, i\%, N) = (1 + i)^N.\)

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Formula for \((P/F, i\%, N)\) (Given \(F\) find \(P\)).

It is reciprocal to \((F/P, i\%, N)\):

\[
(P/F, i\%, N) = \frac{1}{(F/P, i\%, N)} = (1 + i)^{-N}.
\]

Example: How much money you need to put in a bank with annual interest rate 8\% in order to save for $1 million in 30 years time?

\[
P = 1,000,000 \times (P/F, 8\%, 30) = $99,377.
\]
These two formulas involve only a single payment. Next we consider cash flows involving more than two payments.

We start with Uniform Series.

*Formula for* $(F/A, i\%, N)$ (Given annual payment $A$ find $F$):

If one receives $1$ each year, then this accumulates into

$$(F/P, i\%, N - 1) + \cdots + (F/P, i\%, 0)$$

$$= (1 + i)^{N-1} + \cdots + (1 + i)^1 + 1$$

$$= \frac{(1 + i)^N - 1}{i}.$$
Therefore,

\[ (F/A, i\%, N) = \frac{(1 + i)^N - 1}{i}. \]

*Example:* If you deposit $10,000 in a bank account every year for 18 years at interest rate 8%. Then this amount becomes

\[ F = 10,000 \times (F/A, 8\%, 18) = 374,502. \]
Formula for \((A/F, i\%, N)\) (Given \(F\) find \(A\)).

This is simply reciprocal to \((F/A, i\%, N)\), i.e.

\[
(A/F, i\%, N) = \frac{1}{(F/A, i\%, N)} = \frac{i}{(1 + i)^N - 1}.
\]

Example: How much do you need to invest every year in order to yield $1 million in 30 year?

\[
A = 1,000,000 \times (A/F, 8\%, 30) = $8,827.
\]
Formula for \((P/A, i\%, N)\) (Given \(A\) find \(P\)).

We can simply do it in two steps. First we compute \((F/A, i\%, N)\) and then multiple it with \((P/F, i\%, N)\), i.e.,

\[
(P/A, i\%, N) = (F/A, i\%, N) \times (P/F, i\%, N) \n= \frac{(1 + i)^N - 1}{i} \times (1 + i)^{-N} \n= \frac{(1 + i)^N - 1}{i(1 + i)^N} \cdot 
\]

Example: You are running a bank. A customer agrees to pay you $100,000 each year with annual interest rate of 10% for 5 years. How much money will you lend to him?

\[
100,000 \times (P/A, 10\%, 5) = $379,080. 
\]
Formula for \((A/P, i\%, N)\) (Given \(P\) find \(A\)).

It is again reciprocal to \((P/A, i\%, N)\):

\[
(A/P, i\%, N) = \frac{i(1 + i)^N}{(1 + i)^N - 1}.
\]

Example: You want to buy a house at the price of $400,000. You will do this with a mortgage from a bank at the annual interest rate 8% for 30 years. What is your annual payment?

\[
A = 400,000 \times (A/P, 8\%, 30) = 35,531
\]

($2,961 per month).
The annual payment will be less and less as $N$ becomes larger. However, this amount will not go to zero in any way. In fact, if $N$ is sufficiently large, then $A \approx Pi$.

Another interesting point is to understand how much in each annual payment is devoted to interest and how much to principal. It can be calculated that in year $n$, the interest payment is

$$I_n = Pi \frac{(1 + i)^N - (1 + i)^{n-1}}{(1 + i)^N - 1}$$

the payment for reducing the principal is

$$B_n = Pi \frac{(1 + i)^{n-1}}{(1 + i)^N - 1},$$

and the remaining principal is

$$P_n = P \frac{(1 + i)^N - (1 + i)^n}{(1 + i)^N - 1}.$$
More Interest Rate Formulas

The formulas discussed so far can be used to derive other formulas. For example, consider the present worth of deferred annuity:

$$(P/A, i\%, N - J)(P/F, i\%, J)$$

or,

$$(P/A, i\%, N) - (P/A, i\%, J).$$

Example: A father wants to decide how much he has to put in an account which gives 12% annual interest, in order to be able to withdraw $2,000 on his new born son’s 18th, 19th, 20th and 21th birthday?

It is

$$2,000 \times (P/A, 12\%, 4) \times (P/F, 12\%, 17) = 884.46.$$
Example: Suppose that you are 22 years old now. You start investing in a plan with interest rate 10% for 15 years with annual investment $1,000. Then you leave the money in the account until you retire at 65. What will the amount be?

At age 36, the accumulated amount is

\[ 1,000 \times (F/A, 10\%, 15) = $31,772.50 \]

At age 65 this amount becomes

\[ 1,000 \times (F/A, 10\%, 15) \times (F/P, 10\%, 29) = $504,010 \]
Suppose that you have a friend with the same age as you. She decides to delay the investment for 10 years. From age 32 onwards she will start investing $2,000 per year until the retirement at 65. How about her lump sum amount at age 65?
Well, it is

\[ 2,000 \times (F/A, 10\%, 65 - 32 + 1) = 490,953 \]

Moral: Better start saving early!
**Uniform Gradient Series.**

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash Flow</td>
<td>0</td>
<td>$G$</td>
<td>$2G$</td>
<td>...</td>
<td>$(N - 1)G$</td>
</tr>
</tbody>
</table>

*Formula for $(F/G, i\%, N)$ (Given $G$ find $F$).*

We have

\[
(F/G, i\%, N)
= (F/A, i\%, N - 1) + (F/A, i\%, N - 2)
+ \cdots + (F/A, i\%, 1)
= \frac{(1+i)^{N-1} - 1}{i} + \frac{(1+i)^{N-2} - 1}{i} + \cdots + \frac{(1+i)^1 - 1}{i}
= \frac{1}{i} \sum_{n=0}^{N-1} (1+i)^n - \frac{N}{i} = \frac{1}{i^2} [(1+i)^N - 1] - \frac{N}{i}
\]
All other formulas for finding $P$ and $A$ can be derived using the formula above.

**Formula for $(P/G, i\%, N)$ (Given $G$ find $P$).**

\[
(P/G, i\%, N) = (F/G, i\%, N)(P/F, i\%, N)
\]

\[
= \left\{ \frac{1}{i^2}[(1 + i)^N - 1] - \frac{N}{i} \right\} (1 + i)^{-N}
\]

\[
= \frac{1}{i} \left[ \frac{1}{i} - (N + \frac{1}{i}) \frac{1}{(1 + i)^N} \right].
\]
Formula for \((A/G, i\%, N)\) (Given \(G\) find \(A\)).

\[
(A/G, i\%, N) = (F/G, i\%, N)(A/F, i\%, N)
\]

\[
= \left\{ \frac{1}{i^2} \left[ (1 + i)^N - 1 \right] - \frac{N}{i} \right\} \frac{i}{(1 + i)^N - 1}
\]

\[
= \frac{1}{i} - \frac{N}{(1 + i)^N - 1}.
\]
Exercise: Suppose we have a project with the future cash flows as follows:

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash Flow</td>
<td>-8,000</td>
<td>-7,000</td>
<td>-6,000</td>
<td>-5,000</td>
</tr>
</tbody>
</table>

Suppose that the interest rate is 15%. What is the present expense of this project?

\[
P = -8,000(P/A, 15\%, 4) + 1,000(P/G, 15\%, 4)
\]
\[
= -8,000 \times 2.855 + 1,000 \times 3.79
\]
\[
= -19,050
\]
Geometric Gradient Series.

In year \( n + 1 \), the cash flow is

\[ A_{n+1} = A_n(1 + g) \]

where \( g > 0 \) is a given factor.

The present worth of this cash flow is

\[
P = \sum_{n=1}^{N} A_n(1 + i)^{-n} = \sum_{n=1}^{N} A_1(1 + g)^{n-1}(1 + i)^{-n}
\]

\[
= \frac{A_1}{1 + g} \sum_{n=1}^{N} \left( \frac{1 + i}{1 + g} \right)^{-n}.
\]

Therefore the effect of this series is the same as: annuity \( \frac{A_1}{1+g} \) and interest rate \( \frac{1+i}{1+g} - 1 = \frac{i-g}{1+g} \); that is,

\[
\frac{A_1}{1 + g} (P/A, (\frac{i-g}{1+g})\%, N).
\]
Nominal and Effective Interest Rates.

If the compounding period is not a whole year, then it is customary to linearly extend it to an annual rate, known as nominal rate.

Example: money drawn quarterly with interest rate 3% has a nominal annual rate 12%.

The effective rate is: \( (1 + 0.03)^4 - 1 \approx 12.55\% \) per annum.

In general, let \( r \) be the nominal rate, \( M \) be the compounding periods in a year. Then the effective annual rate is

\[
i = (1 + r/M)^M - 1.
\]

Exercise: Show that the continuous rate is an upper bound for the effective annual rate.
Effective versus Nominal Interest Rates

<table>
<thead>
<tr>
<th>N. of C.</th>
<th>6%</th>
<th>10%</th>
<th>15%</th>
<th>24%</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6.09%</td>
<td>10.25%</td>
<td>15.56%</td>
<td>25.44%</td>
</tr>
<tr>
<td>4</td>
<td>6.14%</td>
<td>10.38%</td>
<td>15.87%</td>
<td>26.25%</td>
</tr>
<tr>
<td>12</td>
<td>6.17%</td>
<td>10.47%</td>
<td>16.08%</td>
<td>26.82%</td>
</tr>
<tr>
<td>365</td>
<td>6.18%</td>
<td>10.52%</td>
<td>16.18%</td>
<td>27.11%</td>
</tr>
</tbody>
</table>
Internal rate of return

Given cash flow \((x_0, x_1, ..., x_n)\), the IRR is the interest rate \(r\) satisfying

\[
0 = x_0 + \frac{x_1}{1 + r} + \frac{x_2}{(1 + r)^2} + \cdots + \frac{x_n}{(1 + r)^n}.
\]

**Theorem 1** If the cash flow is such that \(x_0 < 0\) and \(x_i \geq 0\) for \(i = 1, 2, ..., n\), and \(x_1 + x_2 + \cdots + x_n > -x_0\), then there is a unique IRR associated with the cash flow.
• Taxes;
• Inflation;
• Real dollar;
• Actual dollar;
• The correction formula

\[ i_r = \frac{i - f}{1 + f}. \]